

Teaching and Learning Hyperbolic Functions (III); „Integral” Properties And Relations Between Their Inverse

Teodor Dumitru Vălcan¹

Abstract:

In two recent papers with the same generic name as this one and numbered with (I), respectively (II), I presented the definitions, the consequences immediate resulting from these and a series of 92 properties of hyperbolic functions, properties that we divided into seven groups, as follows: A) "Trigonometric" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions - six properties; D) The monotony and the invertibility of hyperbolic functions - 17 properties; E) Other properties "trigonometric" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions - six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. In this paper we will continue this approach and will present and prove another 36 properties of these functions, properties that we will divide into three groups, as follows: H) Properties „integral” and rewithrrrence formulas - 11 properties; I) Relations between the inverse of hyperbolic functions - five properties and K) Relations between the hyperbolic functions and the inverses of other hyperbolic functions - 20 properties.

Key words: hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant.

1. Introduction

As part of a larger project entitled "*Training and developing the competences of children, students and teachers to solve problems / exercises in Mathematics*", in two recent papers with the same generic name as this one and numbered with (I), respectively (II), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "Trigonometric" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions – six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. That in paper (I). In paper (II) I continued this approach and I presented another 54 properties of these functions, properties that have divided into three groups, as follows: E) Other properties "trigonometric" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions – six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. In this paper we will continue this approach and will present and prove another 36 properties of these functions, properties that we will divide into three groups, as follows: H) Properties „integral” and rewithrrrence formulas - 11 properties; I) Relations between the inverse of hyperbolic functions - five properties and K) Relations between the hyperbolic functions and the inverses of other hyperbolic functions - 20 properties.

That is why I ask the reader attentive and interested in these matters to consider this work as a continuation of the two works mentioned above. In this regard we will keep the numbering of the results, thus, the results numbered with (2.1) to (2.16), respectively (3.1) to (3.30) are from work (I) - i.e. (Vălcan, 2016), and those numbered with (4.1) to (4.54') are from work (II) – i.e. (Vălcan, 2019).

These properties, as well as others that we will present and prove later, will be used in various applications in Algebra or Mathematical Analysis.

2. The main results

We present and prove here the 36 properties of the hyperbolic functions, mentioned above.

Proposition: *The following statements hold:*

H. Properties „integral” and rewithrrrence formulas

¹ Didactics of Exact Sciences Department, Babes-Bolyai University, Cluj-Napoca, Romania, tdvalcan@yahoo.ca

1) For every $x \in \mathbf{R}^*$,

$$\int cshx \cdot dx = \ln \left| th \left(\frac{x}{2} \right) \right| + C. \quad (5.1)$$

2) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, if:

$$\alpha_n = \int (shx)^n \cdot dx,$$

then:

$$\alpha_n = \frac{1}{n} \cdot (shx)^{n-1} \cdot chx - \frac{n-1}{n} \cdot \alpha_{n-2}, \quad (5.2)$$

i.e., for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$:

$$\int (shx)^n \cdot dx = \frac{1}{n} \cdot (shx)^{n-1} \cdot chx - \frac{n-1}{n} \cdot \int (shx)^{n-2} \cdot dx. \quad (5.2')$$

Furthermore, for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$,

$$\alpha_{2n+1} = \int (shx)^{2n+1} \cdot dx = \sum_{k=0}^n (-1)^k \cdot \frac{C_n^k}{2n-2k+1} \cdot (chx)^{2n-2k+1} + C. \quad (5.2'')$$

3) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, if:

$$\beta_n = \int (chx)^n \cdot dx,$$

then:

$$\beta_n = \frac{1}{n} \cdot (chx)^{n-1} \cdot shx + \frac{n-1}{n} \cdot \beta_{n-2}, \quad (5.3)$$

i.e., for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$:

$$\int (chx)^n \cdot dx = \frac{1}{n} \cdot (chx)^{n-1} \cdot shx + \frac{n-1}{n} \cdot \int (chx)^{n-2} \cdot dx. \quad (5.3')$$

Furthermore, for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$,

$$\beta_{2n+1} = \int (chx)^{2n+1} \cdot dx = \sum_{k=0}^n \frac{C_n^k}{2n-2k+1} \cdot (shx)^{2n-2k+1} + C. \quad (5.3'')$$

4) For every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}^*$, if:

$$\chi_n = \int (shx)^n \cdot (chx)^{2m+1} \cdot dx,$$

then:

$$\chi_n = \sum_{k=0}^m \frac{C_m^k}{n+2m-2k+1} \cdot (shx)^{n+2m-2k+1} + C. \quad (5.4)$$

5) For every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}^*$, if:

$$\delta_n = \int (shx)^{2n+1} \cdot (chx)^m \cdot dx,$$

then:

$$\delta_n = \sum_{k=0}^m (-1)^k \cdot \frac{C_m^k}{m+2n-2k+1} \cdot (chx)^{m+2n-2k+1} + C. \quad (5.5)$$

6) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\varepsilon_n = \int (thx)^n \cdot dx,$$

then:

$$\varepsilon_n = -\frac{1}{n-1} \cdot (thx)^{n-1} + \varepsilon_{n-2}, \quad (5.6)$$

i.e., for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$:

$$\int (thx)^n \cdot dx = -\frac{1}{n-1} \cdot (thx)^{n-1} + \int (thx)^{n-2} \cdot dx. \quad (5.6')$$

Therefore,

$$\int (thx)^{2n} \cdot dx = x \cdot \sum_{k=1}^n \frac{1}{2k-1} \cdot (thx)^{2k-1} + C \quad (5.6'')$$

and

$$\int (thx)^{2n+1} \cdot dx = \ln(chx) \cdot \sum_{k=1}^n \frac{1}{2k} \cdot (thx)^{2k} + C. \quad (5.6''')$$

7) For every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\phi_n = \int (cthx)^n \cdot dx,$$

then:

$$\phi_n = -\frac{1}{n-1} \cdot (cthx)^{n-1} + \phi_{n-2}, \quad (5.7)$$

i.e., for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$:

$$\int (cthx)^n \cdot dx = -\frac{1}{n-1} \cdot (cthx)^{n-1} + \int (cthx)^{n-2} \cdot dx. \quad (5.7')$$

Therefore,

$$\int (cthx)^{2n} \cdot dx = x \cdot \sum_{k=1}^n \frac{1}{2k-1} \cdot (cthx)^{2k-1} + C \quad (5.7'')$$

and

$$\int (cthx)^{2n+1} \cdot dx = \ln|shx| \cdot \sum_{k=1}^n \frac{1}{2k} \cdot (cthx)^{2k} + C. \quad (5.7''')$$

8) For every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\eta_n = \int (cshx)^n \cdot dx,$$

then:

$$\eta_n = -\frac{1}{n-1} \cdot \frac{chx}{(shx)^{n-1}} - \frac{n-2}{n-1} \cdot \eta_{n-2}, \quad (5.8)$$

i.e., for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$:

$$\int (cshx)^n \cdot dx = -\frac{1}{n-1} \cdot \frac{chx}{(shx)^{n-1}} - \frac{n-2}{n-1} \cdot \int (cshx)^{n-2} \cdot dx. \quad (5.8')$$

Furthermore, for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$,

$$\begin{aligned} \eta_{2n+1} &= \int (cshx)^{2n+1} \cdot dx \\ &= A_1 \cdot \ln|chx-1| + B_1 \cdot \ln(chx+1) - \sum_{k=2}^{n+1} \frac{A_k + B_k}{k-1} \cdot \left[\frac{1}{(chx-1)^{k-1}} + \frac{1}{(chx+1)^{k-1}} \right] + C, \end{aligned} \quad (5.8'')$$

where, for every $k \in \{1, 2, \dots, n+1\}$, A_k and B_k are the coefficients of following developments in simple fractions:

$$\frac{1}{(y^2-1)^{n+1}} = \sum_{k=1}^{n+1} \frac{A_k}{(y-1)^k} + \sum_{k=1}^{n+1} \frac{B_k}{(y+1)^k}.$$

9) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, $n \geq 2$, if:

$$\lambda_n = \int (schx)^n \cdot dx,$$

then:

$$\lambda_n = \frac{1}{n-1} \cdot \frac{shx}{(chx)^{n-1}} + \frac{n-2}{n-1} \cdot \lambda_{n-2}, \quad (5.9)$$

i.e., for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$:

$$\int (schx)^n \cdot dx = \frac{1}{n-1} \cdot \frac{shx}{(chx)^{n-1}} + \frac{n-2}{n-1} \cdot \int (schx)^{n-2} \cdot dx. \quad (5.9')$$

10) For every $x \in \mathbf{R}^*$ and every $m, n \in \mathbf{N}$, with $m \geq 2$ and $n \geq 2$, if:

$$\mu_{n,m} = \int \frac{(chx)^n}{(shx)^m} \cdot dx,$$

then:

$$\mu_{n,m} = \frac{1}{n-m} \cdot \frac{(chx)^{n-1}}{(shx)^{m-1}} + \frac{n-1}{n-m} \cdot \mu_{n-2,m}, \quad (5.10)$$

or

$$\mu_{n,m} = -\frac{1}{m-1} \cdot \frac{(chx)^{n+1}}{(shx)^{m-1}} + \frac{n-m+2}{m-1} \cdot \mu_{n,m-2}, \quad (5.11)$$

or

$$\mu_{n,m} = -\frac{1}{m-1} \cdot \frac{(chx)^{n-1}}{(shx)^{m-1}} + \frac{n-1}{m-1} \cdot \mu_{n-2,m-2}, \quad (5.12)$$

i.e., for every $x \in \mathbf{R}^*$ and every $m, n \in \mathbf{N}$, with $m \geq 2$ and $n \geq 2$:

$$\int \frac{(chx)^n}{(shx)^m} \cdot dx = \frac{1}{n-m} \cdot \frac{(chx)^{n-1}}{(shx)^{m-1}} + \frac{n-1}{n-m} \cdot \int \frac{(chx)^{n-2}}{(shx)^m} \cdot dx, \quad (5.10')$$

or:

$$\int \frac{(chx)^n}{(shx)^m} \cdot dx = -\frac{1}{m-1} \cdot \frac{(chx)^{n+1}}{(shx)^{m-1}} + \frac{n-m+2}{m-1} \cdot \int \frac{(chx)^n}{(shx)^{m-2}} \cdot dx, \quad (5.11')$$

or:

$$\int \frac{(chx)^n}{(shx)^m} \cdot dx = -\frac{1}{m-1} \cdot \frac{(chx)^{n-1}}{(shx)^{m-1}} + \frac{n-1}{m-1} \cdot \int \frac{(chx)^{n-2}}{(shx)^{m-2}} \cdot dx. \quad (5.12')$$

11) For every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}$, $m \geq 2$ and $n \geq 2$, if:

$$\theta_{m,n} = \int \frac{(shx)^m}{(chx)^n} \cdot dx,$$

then:

$$\theta_{m,n} = \frac{1}{m-n} \cdot \frac{(shx)^{m-1}}{(chx)^{n-1}} + \frac{m-1}{n-m} \cdot \theta_{m-2,n}, \quad (5.13)$$

or:

$$\theta_{m,n} = \frac{1}{n-1} \cdot \frac{(shx)^{m+1}}{(chx)^{n-1}} - \frac{m-n+2}{n-1} \cdot \theta_{m,n-2}, \quad (5.14)$$

or:

$$\theta_{m,n} = -\frac{1}{n-1} \cdot \frac{(shx)^{m-1}}{(chx)^{n-1}} + \frac{m-1}{n-1} \cdot \theta_{m-2,n-2}, \quad (5.15)$$

i.e., for every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}$, with $m \geq 2$ and $n \geq 2$:

$$\int \frac{(shx)^m}{(chx)^n} \cdot dx = \frac{1}{m-n} \cdot \frac{(shx)^{m-1}}{(chx)^{n-1}} + \frac{m-1}{n-m} \cdot \int \frac{(shx)^{m-2}}{(chx)^n} \cdot dx, \quad (5.13')$$

or:

$$\int \frac{(shx)^m}{(chx)^n} \cdot dx = \frac{1}{n-1} \cdot \frac{(shx)^{m+1}}{(chx)^{n-1}} + \frac{m-n+2}{n-1} \cdot \int \frac{(shx)^m}{(chx)^{n-2}} \cdot dx, \quad (5.14')$$

or:

$$\int \frac{(shx)^m}{(chx)^n} \cdot dx = -\frac{1}{n-1} \cdot \frac{(shx)^{m-1}}{(chx)^{n-1}} + \frac{m-1}{n-1} \cdot \int \frac{(shx)^{m-2}}{(chx)^{n-2}} \cdot dx. \quad (5.15')$$

I. Relations between the inverse of hyperbolic functions

12) For every $x \in (-\infty, 0)$:

$$sb^{-1}\left(\frac{1}{x}\right) = csh^{-1}x; \quad (5.16)$$

or equivalent:

$$csh^{-1}\left(\frac{1}{x}\right) = sb^{-1}x. \quad (5.16')$$

13) For every $x \in (0, +\infty)$:

$$sb^{-1}\left(\frac{1}{x}\right) = csh^{-1}x; \quad (5.17)$$

or equivalent:

$$csh_2^{-1} \left(\frac{1}{x} \right) = sb^{-1}x. \quad (5.17')$$

14) For every $x \in (0, 1)$,

$$ch_1^{-1} \left(\frac{1}{x} \right) = sch_1^{-1}x; \quad (5.18)$$

or equivalent, for every $x \in (1, +\infty)$:

$$sch_1^{-1} \left(\frac{1}{x} \right) = ch_1^{-1}x. \quad (5.18')$$

15) For every $x \in (0, 1)$,

$$cb_2^{-1} \left(\frac{1}{x} \right) = sch_2^{-1}x; \quad (5.19)$$

or equivalent:

$$sch_2^{-1} \left(\frac{1}{x} \right) = cb_2^{-1}x. \quad (5.19')$$

16) For every $x \in (-1, 1) \setminus \{0\}$,

$$ctb^{-1} \left(\frac{1}{x} \right) = tb^{-1}x; \quad (5.20)$$

or equivalent, for every $x \in (-\infty, -1) \cup (1, +\infty)$:

$$tb^{-1} \left(\frac{1}{x} \right) = ctb^{-1}x. \quad (5.20')$$

J. Relations between the hyperbolic functions and the inverses of other hyperbolic functions

17) For every $x \in [1, +\infty)$,

$$sh(cb_1^{-1}x) = -\sqrt{x^2 - 1} \quad (5.21)$$

and, for every $x \in [1, +\infty)$,

$$sh(cb_2^{-1}x) = \sqrt{x^2 - 1}. \quad (5.21')$$

18) For every $x \in (-1, 1)$,

$$sh(tb^{-1}x) = \frac{|x|}{\sqrt{1 - x^2}}. \quad (5.22)$$

19) For every $x \in (-\infty, -1)$,

$$sh(ctb^{-1}x) = -\frac{1}{\sqrt{x^2 - 1}} \quad (5.23)$$

and, for every $x \in (1, +\infty)$,

$$sh(ctb^{-1}x) = \frac{1}{\sqrt{x^2 - 1}}. \quad (5.23')$$

20) For every $x \in (0, 1)$, if $sch^{-1}x < 0$,

$$sh(sch^{-1}x) = -\frac{\sqrt{1 - x^2}}{x} \quad (5.24)$$

and, for every $x \in (0, 1)$, if $sch^{-1}x > 0$,

$$sh(sch^{-1}x) = \frac{\sqrt{1 - x^2}}{x}. \quad (5.24')$$

21) For every $x \in (-\infty, 0)$,

$$sh(csb_1^{-1}x) = \frac{1}{x} \quad (5.25)$$

and, for every $x \in (0, +\infty)$,

$$sh(csb_2^{-1}x) = \frac{1}{x}. \quad (5.25')$$

22) For every $x \in \mathbf{R}$,

$$cb(sh^{-1}x) = \sqrt{x^2 + 1}. \quad (5.26)$$

23) For every $x \in (-1, 1)$,

$$cb(tb^{-1}x) = \frac{1}{\sqrt{1 - x^2}}. \quad (5.27)$$

24) For every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$cb(ctb^{-1}x) = \frac{|x|}{\sqrt{x^2 - 1}}. \quad (5.28)$$

25) For every $x \in (0, 1)$, if $sch^{-1}x < 0$,

$$cb(sch^{-1}x) = -\frac{1}{x} \quad (5.29)$$

and, for every $x \in (0, 1)$, if $sch^{-1}x > 0$,

$$cb(sch^{-1}x) = \frac{1}{x}. \quad (5.29')$$

26) For every $x \in (-\infty, 0)$,

$$cb(csb_1^{-1}x) = -\frac{\sqrt{x^2 + 1}}{x} \quad (5.30)$$

and, for every $x \in (0, +\infty)$,

$$cb(csb_2^{-1}x) = \frac{\sqrt{x^2 + 1}}{x}. \quad (5.30')$$

27) For every $x \in \mathbf{R}$,

$$tb(sh^{-1}x) = \frac{x}{\sqrt{x^2 + 1}}. \quad (5.31)$$

28) For every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$tb(cb^{-1}x) = \frac{\sqrt{x^2 - 1}}{x}. \quad (5.32)$$

29) For every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$tb(ctb^{-1}x) = \frac{1}{x}. \quad (5.33)$$

30) For every $x \in (0, 1)$, if $sch^{-1}x < 0$,

$$tb(sch^{-1}x) = -\sqrt{1 - x^2} \quad (5.34)$$

and, for every $x \in (0, 1)$, if $sch^{-1}x > 0$,

$$tb(sch^{-1}x) = \sqrt{1 - x^2}. \quad (5.34')$$

31) For every $x \in (-\infty, 0)$,

$$tb(csb_1^{-1}x) = -\frac{1}{\sqrt{x^2 + 1}} \quad (5.35)$$

and, for every $x \in (0, +\infty)$,

$$tb(csb_2^{-1}x) = \frac{1}{\sqrt{x^2 + 1}}. \quad (5.35')$$

32) For every $x \in \mathbf{R}^*$,

$$ctb(sh^{-1}x) = \frac{\sqrt{x^2 + 1}}{x}. \quad (5.36)$$

33) For every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$ctb(ch^{-1}x) = \frac{x}{\sqrt{x^2 - 1}}. \quad (5.37)$$

34) For every $x \in (-1, 1)$,

$$ctb(th^{-1}x) = \frac{1}{x}. \quad (5.38)$$

35) For every $x \in (0, 1)$, if $schr^{-1}x < 0$,

$$ctb(schr^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad (5.39)$$

and, for every $x \in (0, 1)$, if $schr^{-1}x > 0$,

$$ctb(schr^{-1}x) = \frac{1}{\sqrt{1-x^2}}. \quad (5.39')$$

36) For every $x \in (-\infty, 0)$,

$$ctb(csb_1^{-1}x) = -\sqrt{x^2 + 1} \quad (5.40)$$

and, for every $x \in (0, +\infty)$,

$$ctb(csb_2^{-1}x) = \sqrt{x^2 + 1}. \quad (5.40')$$

Proof: 1) For every $x \in \mathbf{R}^*$, the equality (5.1) follows from the equalities (3.21) and (4.9).

2) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, let be:

$$\alpha_n = \int (\sh x)^n \cdot dx,$$

then, according to the equalities (3.11) and (3.1), obtain the equalities:

$$\begin{aligned} (1) \quad \alpha_n &= \int (\sh x)^n \cdot dx = \int \sh x \cdot (\sh x)^{n-1} \cdot dx = \int (\ch x)' \cdot (\sh x)^{n-1} \cdot dx \\ &= \ch x \cdot (\sh x)^{n-1} - \int \ch x \cdot (\sh^{n-1} x)' \cdot dx = \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \int \ch x \cdot (\sh x)^{n-2} \cdot \ch x \cdot dx \\ &= \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \int (\sh x)^{n-2} \cdot \ch^2 x \cdot dx = \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \int (\sh x)^{n-2} \cdot (1 + \sh^2 x) \cdot dx \\ &= \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \int (\sh x)^{n-2} \cdot dx - (n-1) \cdot \int (\sh x)^n \cdot dx \\ &= \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \alpha_{n-2} - (n-1) \cdot \alpha_n. \end{aligned}$$

Retaining the ends of the equalities (1), obtain that:

$$n\alpha_n = \ch x \cdot (\sh x)^{n-1} - (n-1) \cdot \alpha_{n-2},$$

whence it follows that:

$$\alpha_n = \frac{1}{n} \cdot (\sh x)^{n-1} \cdot \ch x - \frac{n-1}{n} \cdot \alpha_{n-2};$$

therefore the equalities (5.2) and (5.2') hold. Furthermore, according to the equality (3.11), for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, obtain that:

$$\begin{aligned} (2) \quad \alpha_{2n+1} &= \int (\sh x)^{2n+1} \cdot dx = \int (\sh x)^{2n} \cdot \sh x \cdot dx = \int (\sh^2 x)^n \cdot \sh x \cdot dx \\ &= \int (\ch^2 x - 1)^n \cdot (\ch x)' \cdot dx. \end{aligned}$$

Making the change of variable:

$$\ch x = y,$$

then:

$$\sh x \cdot dx = dy$$

and must calculate the integral:

$$\begin{aligned} (3) \quad \alpha'_{2n+1} &= \int (y^2 - 1)^n \cdot dy = \int \sum_{k=0}^n (-1)^k \cdot C_n^k \cdot y^{2n-2k} \cdot dy = \sum_{k=0}^n (-1)^k \cdot C_n^k \cdot \int y^{2n-2k} \cdot dy \\ &= \sum_{k=0}^n (-1)^k \cdot \frac{C_n^k}{2n-2k+1} \cdot y^{2n-2k+1} + C. \end{aligned}$$

Returning to the old variable, from the equalities (2) and (3), obtain the equality (5.2'').

3) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, let be:

$$\beta_n = \int (\ch x)^n \cdot dx,$$

then, according to the equalities (3.10) and (3.1), obtain the equalities:

$$\begin{aligned} (1) \quad \beta_n &= \int (\ch x)^n \cdot dx = \int \ch x \cdot (\ch x)^{n-1} \cdot dx = \int (\sh x)' \cdot (\ch x)^{n-1} \cdot dx \\ &= \sh x \cdot (\ch x)^{n-1} - \int \sh x \cdot [(\ch x)^{n-1}]' \cdot dx = \sh x \cdot (\ch x)^{n-1} - (n-1) \cdot \int \sh x \cdot (\ch x)^{n-2} \cdot \sh x \cdot dx \\ &= \sh x \cdot (\ch x)^{n-1} - (n-1) \cdot \int (\ch x)^{n-2} \cdot \sh^2 x \cdot dx = \sh x \cdot (\ch x)^{n-1} - (n-1) \cdot \int (\ch x)^{n-2} \cdot (\ch^2 x - 1) \cdot dx \end{aligned}$$

$$= \text{shx} \cdot (\text{chx})^{n-1} \cdot (n-1) \cdot \int (\text{chx})^n \cdot dx + (n-1) \cdot \int (\text{chx})^{n-2} \cdot dx$$

$$= \text{shx} \cdot (\text{chx})^{n-1} \cdot (n-1) \cdot \beta_n + (n-1) \cdot \beta_{n-2}.$$

Retaining the ends of the equalities (1), obtain that:

$$n\beta_n = \text{shx} \cdot (\text{chx})^{n-1} + (n-1) \cdot \beta_{n-2},$$

whence it follows that:

$$\beta_n = \frac{1}{n} \cdot (\text{chx})^{n-1} \cdot \text{shx} + \frac{n-1}{n} \cdot \beta_{n-2};$$

therefore the equalities (5.3) and (5.3') hold. Furthermore, according to the same equalities (3.10) and (3.1), for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$, obtain that:

$$(2) \beta_{2n+1} = \int (\text{chx})^{2n+1} \cdot dx = \int (\text{chx})^{2n} \cdot \text{chx} \cdot dx = \int (\text{ch}^2 x)^n \cdot \text{chx} \cdot dx \\ = \int (\text{sh}^2 x + 1)^n \cdot (\text{shx})' \cdot dx.$$

Making the change of variable:

$$\text{shx} = y,$$

then:

$$\text{chx} \cdot dx = dy$$

and must calculate the integral:

$$(3) \beta'_{2n+1} = \int (y^2 + 1)^n \cdot dy = \int \sum_{k=0}^n C_n^k y^{2n-2k} \cdot dy = \sum_{k=0}^n C_n^k \cdot \int y^{2n-2k} \cdot dy \\ = \sum_{k=0}^n \frac{C_n^k}{2n-2k+1} \cdot y^{2n-2k+1} + C.$$

Returning to the old variable, from the equalities (2) and (3), obtain the equality (5.3'').

4) For every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}^*$, if:

$$\chi_n = \int (\text{shx})^n \cdot (\text{chx})^{2m+1} \cdot dx,$$

then, according to the equality (3.1):

$$(1) \chi_n = \int (\text{shx})^n \cdot (\text{chx})^{2m+1} \cdot dx = \int (\text{shx})^n \cdot (\text{chx})^{2m} \cdot \text{chx} \cdot dx \\ = \int (\text{shx})^n \cdot (1 + \text{sh}^2 x)^m \cdot \text{chx} \cdot dx.$$

Making the change of variable:

$$\text{shx} = y,$$

then:

$$\text{chx} \cdot dx = dy$$

and must calculate the integral:

$$(2) \chi'_{2n+1} = \int y^n \cdot (y^2 + 1)^m \cdot dy = \int y^n \cdot \sum_{k=0}^m C_m^k y^{2m-2k} \cdot dy = \sum_{k=0}^m C_m^k \cdot \int y^{n+2m-2k} \cdot dy \\ = \sum_{k=0}^m \frac{C_m^k}{n+2m-2k+1} \cdot y^{n+2m-2k+1} + C.$$

Returning to the old variable, from the equalities (1) and (2), obtain the equality (5.4).

5) For every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}^*$, if:

$$\delta_n = \int (\text{shx})^{2n+1} \cdot (\text{chx})^m \cdot dx,$$

then, according to the equality (3.1):

$$(1) \delta_n = \int (\text{shx})^{2n+1} \cdot (\text{chx})^m \cdot dx = \int (\text{shx})^{2n} \cdot (\text{chx})^m \cdot \text{shx} \cdot dx \\ = \int (\text{chx})^m \cdot (\text{ch}^2 x - 1)^n \cdot \text{shx} \cdot dx.$$

Making the change of variable:

$$\text{chx} = y,$$

then:

$$\text{shx} \cdot dx = dy$$

and must calculate the integral:

$$(2) \delta'_{2n+1} = \int y^m \cdot (y^2 - 1)^n \cdot dy = \int y^m \cdot \sum_{k=0}^n (-1)^k \cdot C_n^k \cdot y^{2n-2k} \cdot dy = \sum_{k=0}^n (-1)^k \cdot C_n^k \cdot \int y^{m+2n-2k} \cdot dy$$

$$= \sum_{k=0}^m (-1)^k \cdot \frac{C_m^k}{m+2n-2k+1} \cdot y^{m+2n-2k+1} + C.$$

Returning to the old variable, from the equalities (1) and (2), obtain the equality (5.5).

6) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\varepsilon_n = \int (\text{th}x)^n \cdot dx,$$

then, according to the equalities (4.21) and (3.12), obtain that:

$$\begin{aligned} (1) \quad \varepsilon_n &= \int (\text{th}x)^n \cdot dx = \int (\text{th}x)^{n-2} \cdot \text{th}^2 x \cdot dx = - \int (\text{th}x)^{n-2} \cdot (1 - \text{th}^2 x - 1) \cdot dx \\ &= - \int (\text{th}x)^{n-2} \cdot (1 - \text{th}^2 x) \cdot dx + \int (\text{th}x)^{n-2} \cdot dx = - \int (\text{th}x)^{n-2} \cdot (\text{th}x)' \cdot dx + \varepsilon_{n+2} \\ &= - \frac{1}{n-1} \cdot (\text{th}x)^{n-1} + \varepsilon_{n-2}. \end{aligned}$$

Therefore, for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$, the equalities (5.6) and (5.6') hold. Furthermore, from the equality (5.6), it follows that:

$$\begin{aligned} \varepsilon_{2n} - \varepsilon_{2n-2} &= - \frac{1}{2n-1} \cdot (\text{th}x)^{2n-1}, \\ \varepsilon_{2n-2} - \varepsilon_{2n-4} &= - \frac{1}{2n-3} \cdot (\text{th}x)^{2n-3}, \\ &\dots\dots\dots \\ \varepsilon_4 - \varepsilon_2 &= - \frac{1}{3} \cdot \text{th}^3 x, \\ \varepsilon_2 - \varepsilon_0 &= - \text{th}x. \end{aligned}$$

By summing, member with member, of these equalities, taking into account the fact that:

$$\varepsilon_0 = \int dx = x + C,$$

obtain that:

$$\varepsilon_{2n} = x - \sum_{k=1}^n \frac{1}{2k-1} \cdot (\text{th}x)^{2k-1} + C;$$

i.e. the equality (5.6'') holds. Proceeding analogously, also from the equality (5.6), obtain that:

$$\begin{aligned} \varepsilon_{2n+1} - \varepsilon_{2n-1} &= - \frac{1}{2n} \cdot (\text{th}x)^{2n}, \\ \varepsilon_{2n-1} - \varepsilon_{2n-3} &= - \frac{1}{2n-2} \cdot (\text{th}x)^{2n-2}, \\ &\dots\dots\dots \\ \varepsilon_5 - \varepsilon_3 &= - \frac{1}{4} \cdot \text{th}^4 x, \\ \varepsilon_3 - \varepsilon_1 &= - \frac{1}{2} \cdot \text{th}^2 x. \end{aligned}$$

Also in this case, by summing, member with member, of these equalities, taking into account the fact that, according to the equality (3.18):

$$\varepsilon_1 = \int \text{th}x \cdot dx = \ln(\text{ch}x)_x + C,$$

we obtain that:

$$\varepsilon_{2n+1} = \ln(\text{ch}x) - \sum_{k=1}^n \frac{1}{2k} \cdot (\text{th}x)^{2k} + C;$$

i.e. the equality (5.6''') holds also.

7) For every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\phi_n = \int (\text{cth}x)^n \cdot dx,$$

then, according to the equalities (4.22) and (3.13), obtain that:

$$\begin{aligned} \phi_n &= \int (\operatorname{cthx})^n \cdot dx = \int (\operatorname{cthx})^{n-2} \cdot \operatorname{cth}^2 x \cdot dx = - \int (\operatorname{cthx})^{n-2} \cdot (1 - \operatorname{cth}^2 x - 1) \cdot dx \\ &= - \int (\operatorname{cthx})^{n-2} \cdot (1 - \operatorname{cth}^2 x) \cdot dx + \int (\operatorname{cthx})^{n-2} \cdot dx = - \int (\operatorname{cthx})^{n-2} \cdot (\operatorname{cthx})' \cdot dx + \phi_{n+2} \\ &= - \frac{1}{n-1} \cdot (\operatorname{cthx})^{n-1} + \phi_{n-2}. \end{aligned}$$

Therefore, for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, the equalities (5.7) and (5.7') hold. Furthermore, from the equality (5.7), it follows that:

$$\begin{aligned} \phi_{2n} - \phi_{2n-2} &= - \frac{1}{2n-1} \cdot (\operatorname{cthx})^{2n-1}, \\ \phi_{2n-2} - \phi_{2n-4} &= - \frac{1}{2n-3} \cdot (\operatorname{cthx})^{2n-3}, \\ &\dots\dots\dots \\ \phi_4 - \phi_2 &= - \frac{1}{3} \cdot \operatorname{cth}^3 x, \\ \phi_2 - \phi_0 &= - \operatorname{cthx}. \end{aligned}$$

By summing, member with member, of these equalities, taking into account the fact that:

$$\phi_0 = \int dx = x + C,$$

obtain that:

$$\phi_{2n} = x - \sum_{k=1}^n \frac{1}{2k-1} \cdot (\operatorname{cthx})^{2k-1} + C;$$

i.e. the equality (5.7'') holds. Proceeding analogously, again from the equality (5.7), obtain that:

$$\begin{aligned} \phi_{2n+1} - \phi_{2n-1} &= - \frac{1}{2n} \cdot (\operatorname{cthx})^{2n}, \\ \phi_{2n-1} - \phi_{2n-3} &= - \frac{1}{2n-2} \cdot (\operatorname{cthx})^{2n-2}, \\ &\dots\dots\dots \\ \phi_5 - \phi_3 &= - \frac{1}{4} \cdot \operatorname{cth}^4 x, \\ \phi_3 - \phi_1 &= - \frac{1}{2} \cdot \operatorname{cth}^2 x. \end{aligned}$$

Also in this case, by summing, member with member, of these equalities, taking into account the fact that, according to the equality (3.19):

$$\phi_1 = \int \operatorname{cthx} \cdot dx = \ln|\operatorname{shx}| + C,$$

we obtain that:

$$\phi_{2n+1} = \ln|\operatorname{shx}| - \sum_{k=1}^n \frac{1}{2k} \cdot (\operatorname{cthx})^{2k} + C;$$

i.e. the equality (5.7''') holds also.

8) For every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\eta_n = \int (\operatorname{cshx})^n \cdot dx,$$

then, according to the equalities (2.6) and (3.1), we have the equalities:

$$\begin{aligned} \text{(1) } \eta_n &= \int (\operatorname{cshx})^n \cdot dx = \int \frac{1}{(\operatorname{shx})^n} \cdot dx = \int \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{(\operatorname{shx})^n} \cdot dx = \int \frac{\operatorname{ch}^2 x}{(\operatorname{shx})^n} \cdot dx - \int \frac{1}{(\operatorname{shx})^{n-2}} \cdot dx \\ &= \int \frac{\operatorname{ch}^2 x}{(\operatorname{shx})^n} \cdot dx - \eta_{n-2}. \end{aligned}$$

Now, according to the equality (3.10), calculating by parts, integral below, obtain that:

$$\begin{aligned}
 (2) \quad \eta'_n &= \int \frac{\operatorname{ch}^2 x}{(\operatorname{sh} x)^n} \cdot dx = \int \frac{\operatorname{ch} x}{(\operatorname{sh} x)^n} \cdot \operatorname{ch} x \cdot dx = \int \left[-\frac{1}{n-1} \cdot \frac{1}{(\operatorname{sh} x)^{n-1}} \right]' \cdot \operatorname{ch} x \cdot dx \\
 &= -\frac{1}{n-1} \cdot \frac{\operatorname{ch} x}{(\operatorname{sh} x)^{n-1}} + \frac{1}{n-1} \cdot \int \frac{\operatorname{sh} x}{(\operatorname{sh} x)^{n-1}} dx = -\frac{1}{n-1} \cdot \frac{\operatorname{ch} x}{(\operatorname{sh} x)^{n-1}} + \frac{1}{n-1} \cdot \int \frac{1}{(\operatorname{sh} x)^{n-2}} dx \\
 &= -\frac{1}{n-1} \cdot \frac{\operatorname{ch} x}{(\operatorname{sh} x)^{n-1}} + \frac{1}{n-1} \cdot \eta_{n-2}.
 \end{aligned}$$

From the equalities (1) and (2), it follows that:

$$\eta_n = -\frac{1}{n-1} \cdot \frac{\operatorname{ch} x}{(\operatorname{sh} x)^{n-1}} - \frac{n-2}{n-1} \cdot \eta_{n-2},$$

i.e., for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, the equalities (5.8) and (5.8') hold. Furthermore, for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$,

$$\begin{aligned}
 (3) \quad \eta_{2n+1} &= \int (\operatorname{csh} x)^{2n+1} \cdot dx = \int \frac{1}{(\operatorname{sh} x)^{2n+1}} \cdot dx = \int \frac{\operatorname{sh} x}{(\operatorname{sh} x)^{2n+2}} \cdot dx = \int \frac{(\operatorname{ch} x)'}{(\operatorname{sh}^2 x)^{n+1}} \cdot dx \\
 &= \int \frac{(\operatorname{ch} x)'}{(\operatorname{ch}^2 x - 1)^{n+1}} \cdot dx.
 \end{aligned}$$

By carrying out the change of variable:

$$\operatorname{ch} x = y,$$

then:

$$(\operatorname{ch} x)' \cdot dx = dy$$

and the integral (3) becomes:

$$(4) \quad \eta'_{2n+1} = \int \frac{1}{(y^2 - 1)^{n+1}} \cdot dx.$$

If:

$$(5) \quad \frac{1}{(y^2 - 1)^{n+1}} = \sum_{k=1}^{n+1} \frac{A_k}{(y-1)^k} + \sum_{k=1}^{n+1} \frac{B_k}{(y+1)^k}$$

is decomposition in simple fractions of expression under integral η'_{2n+1} , then, from the equality (4), it follows that:

$$(6) \quad \eta'_{2n+1} = A_1 \cdot \ln|y-1| + B_1 \cdot \ln|y+1| - \sum_{k=2}^{n+1} \frac{A_k + B_k}{k-1} \cdot \left[\frac{1}{(y-1)^{k-1}} + \frac{1}{(y+1)^{k-1}} \right] + C.$$

Now, from the equalities (3) and (6), obtain the equality (5.8'').

9) For every $x \in \mathbf{R}$ and every $n \in \mathbf{N}$, with $n \geq 2$, if:

$$\lambda_n = \int (\operatorname{sch} x)^n \cdot dx,$$

then, according to the equalities (2.5) and (3.1), we have the equalities:

$$\begin{aligned}
 (1) \quad \lambda_n &= \int (\operatorname{sch} x)^n \cdot dx = \int \frac{1}{(\operatorname{ch} x)^n} \cdot dx = \int \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{(\operatorname{ch} x)^n} \cdot dx = \int \frac{1}{(\operatorname{ch} x)^{n-2}} \cdot dx - \int \frac{\operatorname{sh}^2 x}{(\operatorname{ch} x)^n} \cdot dx \\
 &= \lambda_{n-2} - \int \frac{\operatorname{sh}^2 x}{(\operatorname{ch} x)^n} \cdot dx.
 \end{aligned}$$

Now, according to the equality (3.11), calculating by parts, integral below, obtain that:

$$\begin{aligned}
 (2) \quad \lambda'_n &= \int \frac{\operatorname{sh}^2 x}{(\operatorname{ch} x)^n} \cdot dx = \int \frac{\operatorname{sh} x}{(\operatorname{ch} x)^n} \cdot \operatorname{sh} x \cdot dx = \int \left[-\frac{1}{n-1} \cdot \frac{1}{(\operatorname{ch} x)^{n-1}} \right]' \cdot \operatorname{sh} x \cdot dx \\
 &= -\frac{1}{n-1} \cdot \frac{\operatorname{sh} x}{(\operatorname{ch} x)^{n-1}} + \frac{1}{n-1} \cdot \int \frac{\operatorname{ch} x}{(\operatorname{ch} x)^{n-1}} dx = -\frac{1}{n-1} \cdot \frac{\operatorname{sh} x}{(\operatorname{ch} x)^{n-1}} + \frac{1}{n-1} \cdot \int \frac{1}{(\operatorname{ch} x)^{n-2}} dx \\
 &= -\frac{1}{n-1} \cdot \frac{\operatorname{sh} x}{(\operatorname{ch} x)^{n-1}} + \frac{1}{n-1} \cdot \lambda_{n-2}.
 \end{aligned}$$

From the equalities (1) and (2), it follows that:

$$\lambda_n = \frac{1}{n-1} \cdot \frac{\operatorname{sh} x}{(\operatorname{ch} x)^{n-1}} + \frac{n-2}{n-1} \cdot \lambda_{n-2},$$

i.e., for every $x \in \mathbf{R}^*$ and every $n \in \mathbf{N}$, with $n \geq 2$, the equalities (5.9) and (5.9') hold.

10) For every $x \in \mathbf{R}^*$, let be:

$$\mu_{n,m} = \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx,$$

where $m, n \in \mathbf{N}$, $m \geq 2$ and $n \geq 2$. First we observe that, for every $x \in \mathbf{R}^*$ and every $m, n \in \mathbf{N}$, with $m \geq 2$ and $n \geq 2$, according to the equality (3.1),

$$\begin{aligned} (1) \quad \mu_{n,m} &= \int \frac{(\operatorname{ch}x)^{n-2} \cdot \operatorname{ch}^2x}{(\operatorname{sh}x)^m} \cdot dx = \int \frac{(\operatorname{ch}x)^{n-2} \cdot (1 + \operatorname{sh}^2x)}{(\operatorname{sh}x)^m} \cdot dx \\ &= \int \left[\frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^m} + \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} \right] \cdot dx = \int \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^m} \cdot dx + \int \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} \cdot dx \\ &= \mu_{n-2,m} + \mu_{n-2,m-2}. \end{aligned}$$

Now, because:

$$\begin{aligned} (2) \quad \left[\frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} \right]' &= \frac{(n-1) \cdot (\operatorname{ch}x)^{n-2} \cdot \operatorname{sh}x \cdot (\operatorname{sh}x)^{m-1} - (m-1) \cdot (\operatorname{ch}x)^{n-1} \cdot \operatorname{ch}x \cdot (\operatorname{sh}x)^{m-2}}{(\operatorname{sh}x)^{2m-2}} \\ &= (n-1) \cdot \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} - (m-1) \cdot \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m}, \end{aligned}$$

integration by parts the equality (2), and taking into account the equality (1), obtain that:

$$\begin{aligned} (3) \quad \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} &= (n-1) \cdot \int \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} \cdot dx - (m-1) \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx \\ &= (n-1) \cdot \mu_{n-2,m-2} - (m-1) \cdot \mu_{n,m} = (n-1) \cdot (\mu_{n,m} - \mu_{n-2,m}) - (m-1) \cdot \mu_{n,m} \\ &= (n-m) \cdot \mu_{n,m} - (n-1) \cdot \mu_{n-2,m}. \end{aligned}$$

From the equality (3) it follows, immediately, the equalities (5.10), respective (5.10'). On the other hand, according to the equalities (3.13) and (3.1), integration by parts, obtain that:

$$\begin{aligned} \mu_{n,m} &= \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx = \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot \frac{1}{\operatorname{sh}^2x} \cdot dx = \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot (-\operatorname{cth}x)' \cdot dx \\ &= -\frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot \operatorname{cth}x + \int \left[\frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \right]' \cdot \operatorname{cth}x \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + \int \frac{n \cdot (\operatorname{ch}x)^{n-1} \cdot \operatorname{sh}x \cdot (\operatorname{sh}x)^{m-2} - (m-2) \cdot (\operatorname{ch}x)^n \cdot \operatorname{ch}x \cdot (\operatorname{sh}x)^{m-3}}{(\operatorname{sh}x)^{2m-4}} \cdot \operatorname{cth}x \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + \int \left[\frac{n \cdot (\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-3}} - \frac{(m-2) \cdot (\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} \right] \cdot \frac{\operatorname{ch}x}{\operatorname{sh}x} \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + n \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot dx - (m-2) \cdot \int \frac{(\operatorname{ch}x)^{n+2}}{(\operatorname{sh}x)^m} \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + n \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot dx - (m-2) \cdot \int \frac{(\operatorname{ch}x)^n \cdot (1 + \operatorname{sh}^2x)}{(\operatorname{sh}x)^m} \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + (n-m+2) \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^{m-2}} \cdot dx - (m-2) \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx \\ &= -\frac{(\operatorname{ch}x)^{n+1}}{(\operatorname{sh}x)^{m-1}} + (n-m+2) \cdot \mu_{n,m-2} - (m-2) \cdot \mu_{n,m}; \end{aligned}$$

whence it follows that the equalities (5.11), respective (5.11') hold. Finally, from the equality (3.10), using (also) integration by parts, obtain:

$$\mu_{n,m} = \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx = \int \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^m} \cdot \operatorname{ch}x \cdot dx = \int \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^m} \cdot (\operatorname{sh}x)' \cdot dx$$

$$\begin{aligned}
&= \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^m} \cdot \operatorname{sh}x - \int \left[\frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^m} \right]' \cdot \operatorname{sh}x \cdot dx \\
&= \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} - \int \frac{(n-1) \cdot (\operatorname{ch}x)^{n-2} \cdot \operatorname{sh}x \cdot (\operatorname{sh}x)^m - m \cdot (\operatorname{ch}x)^{n-1} \cdot \operatorname{ch}x \cdot (\operatorname{sh}x)^{m-1}}{(\operatorname{sh}x)^{2m}} \cdot \operatorname{sh}x \cdot dx \\
&= \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} - \int \left[\frac{(n-1) \cdot (\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} - \frac{m \cdot (\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \right] \cdot dx \\
&= \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} - (n-1) \cdot \int \frac{(\operatorname{ch}x)^{n-2}}{(\operatorname{sh}x)^{m-2}} \cdot dx + m \cdot \int \frac{(\operatorname{ch}x)^n}{(\operatorname{sh}x)^m} \cdot dx \\
&= \frac{(\operatorname{ch}x)^{n-1}}{(\operatorname{sh}x)^{m-1}} - (n-1) \cdot \mu_{n-2, m-2} + m \cdot \mu_{n, m};
\end{aligned}$$

whence it follows that the equalities (5.12), respective (5.12') hold.

11) For every $x \in \mathbf{R}$, let be:

$$\theta_{m,n} = \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx,$$

where $m, n \in \mathbf{N}$, $m \geq 2$ and $n \geq 2$. First we observe that, for every $x \in \mathbf{R}$ and every $m, n \in \mathbf{N}$, with $m \geq 2$ and $n \geq 2$, According to the equality (3.1),

$$\begin{aligned}
(1) \quad \theta_{m,n} &= \int \frac{(\operatorname{sh}x)^{m-2} \cdot \operatorname{sh}^2x}{(\operatorname{ch}x)^n} \cdot dx = \int \frac{(\operatorname{sh}x)^{m-2} \cdot (\operatorname{ch}^2x - 1)}{(\operatorname{ch}x)^n} \cdot dx \\
&= \int \left[\frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^{n-2}} - \frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^n} \right] \cdot dx = \int \frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^{n-2}} \cdot dx - \int \frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^n} \cdot dx \\
&= \theta_{m-2, n-2} - \theta_{m-2, n}.
\end{aligned}$$

Now, because:

$$\begin{aligned}
(2) \quad \left[\frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} \right]' &= \frac{(m-1) \cdot (\operatorname{sh}x)^{m-2} \cdot \operatorname{ch}x \cdot (\operatorname{ch}x)^{n-1} - (n-1) \cdot (\operatorname{sh}x)^{m-1} \cdot \operatorname{sh}x \cdot (\operatorname{ch}x)^{n-2}}{(\operatorname{ch}x)^{2n-2}} \\
&= (m-1) \cdot \frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^{n-2}} - (n-1) \cdot \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n},
\end{aligned}$$

integration the equality (2), and taking into account the equality (1), obtain that:

$$\begin{aligned}
(3) \quad \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} &= (m-1) \cdot \int \frac{(\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^{n-2}} \cdot dx - (n-1) \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx \\
&= (m-1) \cdot \theta_{m-2, n-2} - (n-1) \cdot \theta_{m,n} = (m-1) \cdot (\theta_{m,n} + \theta_{m-2, n}) - (n-1) \cdot \theta_{m,n} \\
&= (m-n) \cdot \theta_{m,n} + (m-1) \cdot \theta_{m-2, n}.
\end{aligned}$$

From the equality (3) it follows, immediately, the equalities (5.13), respective (5.13'). On the other hand, according to the equalities (3.12) and (3.1), integration by parts, obtain that:

$$\begin{aligned}
\theta_{m,n} &= \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx = \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot \frac{1}{\operatorname{ch}^2x} \cdot dx = \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot (\operatorname{th}x)' \cdot dx \\
&= \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot \operatorname{th}x - \int \left[\frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \right]' \cdot \operatorname{th}x \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - \int \frac{m \cdot (\operatorname{sh}x)^{m-1} \cdot \operatorname{ch}x \cdot (\operatorname{ch}x)^{n-2} - (n-2) \cdot (\operatorname{sh}x)^m \cdot \operatorname{sh}x \cdot (\operatorname{ch}x)^{n-3}}{(\operatorname{ch}x)^{2n-4}} \cdot \operatorname{th}x \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - \int \left(\frac{m \cdot (\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-3}} - \frac{(n-2) \cdot (\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} \right) \cdot \frac{\operatorname{sh}x}{\operatorname{ch}x} \cdot dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - m \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot dx + (n-2) \cdot \int \frac{(\operatorname{sh}x)^{m+2}}{(\operatorname{ch}x)^n} \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - m \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot dx + (n-2) \cdot \int \frac{(\operatorname{sh}x)^m \cdot (\operatorname{ch}^2 x - 1)}{(\operatorname{ch}x)^n} \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - m \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot dx - (n-2) \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx + (n-2) \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - (m-n+2) \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^{n-2}} \cdot dx - (n-2) \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m+1}}{(\operatorname{ch}x)^{n-1}} - (m-n+2) \cdot \theta_{m,n-2} - (n-2) \cdot \theta_{m,n};
\end{aligned}$$

whence it follows that the equalities (5.14), respective (5.14') hold. Finally, from the equality (3.11), using (also) integration by parts, obtain:

$$\begin{aligned}
\theta_{m,n} &= \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx = \int \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^n} \cdot \operatorname{sh}x \cdot dx = \int \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^n} \cdot (\operatorname{ch}x)' \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^n} \cdot \operatorname{ch}x - \int \left[\frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^n} \right]' \cdot \operatorname{ch}x \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} - \int \frac{(m-1) \cdot (\operatorname{sh}x)^{m-2} \cdot \operatorname{ch}x \cdot (\operatorname{ch}x)^n - n \cdot (\operatorname{sh}x)^{m-1} \cdot \operatorname{sh}x \cdot (\operatorname{ch}x)^{n-1}}{(\operatorname{ch}x)^{2n}} \cdot \operatorname{ch}x \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} - \int \left(\frac{(m-1) \cdot (\operatorname{sh}x)^{m-2}}{(\operatorname{ch}x)^{n-2}} - \frac{n \cdot (\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \right) \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} - (m-1) \cdot \int \frac{(\operatorname{ch}x)^{m-2}}{(\operatorname{sh}x)^{n-2}} \cdot dx + n \cdot \int \frac{(\operatorname{sh}x)^m}{(\operatorname{ch}x)^n} \cdot dx \\
&= \frac{(\operatorname{sh}x)^{m-1}}{(\operatorname{ch}x)^{n-1}} - (m-1) \cdot \theta_{m-2,n-2} + n \cdot \theta_{m,n};
\end{aligned}$$

whence it follows that the equalities (5.15), respective (5.15') hold.

12) According to the equalities (3.22) and (3.29), for every $x \in (-\infty, 0)$:

$$\begin{aligned}
\operatorname{sh}^{-1}\left(\frac{1}{x}\right) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2 + 1}}{|x|}\right) = \ln\left(\frac{1}{x} - \frac{\sqrt{x^2 + 1}}{x}\right) \\
&= \ln\left(\frac{1 - \sqrt{x^2 + 1}}{x}\right) \\
&= \operatorname{csh}_1^{-1} x;
\end{aligned}$$

so, the equalities (5.16) and (by default!) (5.16') hold.

13) According to the equalities (3.22) and (3.30), for every $x \in (0, +\infty)$:

$$\begin{aligned}
\operatorname{sh}^{-1}\left(\frac{1}{x}\right) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2 + 1}}{|x|}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2 + 1}}{x}\right) \\
&= \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right) \\
&= \operatorname{csh}_2^{-1} x;
\end{aligned}$$

so, the equalities (5.17) and (by default!) (5.17') hold.

14) According to the equalities (3.23) and (3.27), for every $x \in (0, 1)$,

$$\begin{aligned}\operatorname{ch}_1^{-1}\left(\frac{1}{x}\right) &= \ln\left(\frac{1}{x} - \sqrt{\frac{1}{x^2} - 1}\right) = \ln\left(\frac{1}{x} - \frac{\sqrt{1-x^2}}{|x|}\right) = \ln\left(\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right) \\ &= \ln\left(\frac{1 - \sqrt{1-x^2}}{x}\right) \\ &= \operatorname{sch}_1^{-1} x;\end{aligned}$$

so the equalities (5.18) and (by default!) (5.18') hold, because, in this case, $\frac{1}{x} \in (1, +\infty)$.

15) According to the equalities (3.24) and (3.28), for every $x \in (0, 1)$,

$$\begin{aligned}\operatorname{ch}_2^{-1}\left(\frac{1}{x}\right) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{|x|}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right) \\ &= \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right) \\ &= \operatorname{sch}_2^{-1} x;\end{aligned}$$

so, the equalities (5.19) and (by default!) (5.19') hold.

16) According to the equalities (3.25) and (3.26''), for every $x \in (-\infty, -1) \cup (1, +\infty)$:

$$\begin{aligned}\operatorname{th}^{-1}\left(\frac{1}{x}\right) &= \frac{1}{2} \cdot \ln \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{1}{2} \cdot \ln \frac{x+1}{x-1} \\ &= \operatorname{cth}^{-1}(x);\end{aligned}$$

so, the equalities (5.20') and (by default!) (5.20) hold, because, in this case, $x \in (-1, 1) \setminus \{0\}$.

17) According to the equalities (3.1) and (3.23), for every $x \in (1, +\infty)$,

(1) $\operatorname{ch}^2(\operatorname{ch}_1^{-1} x) - \operatorname{sh}^2(\operatorname{ch}_1^{-1} x) = 1$,

i.e.:

(2) $x^2 - \operatorname{sh}^2(\operatorname{ch}_1^{-1} x) = 1$.

From the equality (2), according to Proposition 3.1, point 25), obtain the equality (5.21). In an analogous manner, according to the equalities (3.1) and (3.24), for every $x \in [1, +\infty)$,

(3) $\operatorname{ch}^2(\operatorname{ch}_2^{-1} x) - \operatorname{sh}^2(\operatorname{ch}_2^{-1} x) = 1$,

i.e.:

(4) $x^2 - \operatorname{sh}^2(\operatorname{ch}_2^{-1} x) = 1$.

From the equality (4), according to Proposition 3.1, point 26), obtain the equality (5.21').

18) According to the equalities (4.21) and (3.1), for every $x \in (-1, 1)$,

(1) $1 - \operatorname{th}^2(\operatorname{th}^{-1} x) = \frac{1}{\operatorname{ch}^2(\operatorname{th}^{-1} x)}$
 $= \frac{1}{1 + \operatorname{sh}^2(\operatorname{th}^{-1} x)},$

i.e.:

(2) $1 + \operatorname{sh}^2(\operatorname{th}^{-1} x) = \frac{1}{1 - x^2}.$

From the equality (2), it follows that:

$$\operatorname{sh}^2(\operatorname{th}^{-1} x) = \frac{x^2}{1 - x^2},$$

whence obtain the equality (5.22).

19) According to the equality (4.22''), for every $x \in (-\infty, -1)$,

$$(1) \operatorname{sh}^2(\operatorname{cth}^{-1}x) = \frac{1}{\operatorname{cth}^2(\operatorname{cth}^{-1}x) - 1},$$

i.e.:

$$(2) \operatorname{sh}^2(\operatorname{cth}^{-1}x) = \frac{1}{x^2 - 1}.$$

From the equality (2), according to Proposition 3.1, point 30), obtain the equality (5.23), and according to Proposition 3.1, point 31), obtain the equality (5.23').

20) According to the equalities (3.1) and (4.21'') (or to the equality (2.5)), for every $x \in (0,1)$,

$$(1) \operatorname{ch}^2(\operatorname{sch}^{-1}x) = \frac{1}{\operatorname{sch}^2(\operatorname{sch}^{-1}x)},$$

i.e.:

$$(2) 1 + \operatorname{sh}^2(\operatorname{sch}^{-1}x) = \frac{1}{x^2}.$$

From the equality (2), according to Proposition 3.1, point 34) and to the equality (2.1), obtain the equality (5.24), and, according to Proposition 3.1, point 35) and the same equality (2.1), obtain the equality (5.24').

21) According to the equality (4.22'') or to the equality (2.6), for every $x \in (-\infty, 0)$,

$$(1) \operatorname{sh}^2(\operatorname{csh}_1^{-1}x) = \frac{1}{\operatorname{csh}_1^2(\operatorname{csh}_1^{-1}x)} \\ = \frac{1}{x^2};$$

whence, according to Proposition 3.1, point 37) and to the equality (2.1), obtain the equality (5.25), and, according to Proposition 3.1, point 38) and the same equality (2.1), obtain the equality (5.25').

22) According to the equality (3.1), for every $x \in \mathbf{R}$, we have:

$$(1) \operatorname{ch}^2(\operatorname{sh}^{-1}x) = 1 + \operatorname{sh}^2(\operatorname{sh}^{-1}x),$$

i.e.:

$$(2) \operatorname{ch}^2(\operatorname{sh}^{-1}x) = 1 + x^2.$$

From the equality (2), obtain the equality (5.26).

23) According to the first equality from (4.21), for every $x \in (-1, 1)$, we have:

$$(1) \operatorname{ch}^2(\operatorname{th}^{-1}x) = \frac{1}{1 - \operatorname{th}^2(\operatorname{th}^{-1}x)} \\ = \frac{1}{1 - x^2}.$$

From the equality (1) and Definition 2.2, obtain the equality (5.27).

24) According to the equality (4.21'') and Definition 2.4, for every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$(1) \operatorname{ch}^2(\operatorname{cth}^{-1}x) = \frac{1}{1 - \operatorname{th}^2(\operatorname{cth}^{-1}x)} = \frac{1}{1 - \frac{1}{\operatorname{cth}^2(\operatorname{cth}^{-1}x)}} = \frac{1}{1 - \frac{1}{x^2}} \\ = \frac{x^2}{x^2 - 1}.$$

From the equality (1) and Definition 2.2, obtain the equality (5.28).

25) According to the equalities (3.1) and (4.21'') (or to the equality (2.5)), for every $x \in (0,1)$,

$$(1) \operatorname{ch}^2(\operatorname{sch}^{-1}x) = \frac{1}{\operatorname{sch}^2(\operatorname{sch}^{-1}x)},$$

i.e.:

$$(2) \operatorname{ch}^2(\operatorname{sch}^{-1}x) = \frac{1}{x^2}.$$

From the equality (2), according to Proposition 3.1, point 34) and to the equality (2.2), obtain the equality (5.29), and, according to Proposition 3.1, point 35) and the same equality (2.2), obtain the equality (5.29').

26) According to the equalities (3.1) and (4.22'') (or to the equality (2.6)), for every $x \in (-\infty, 0)$,

$$(1) \operatorname{sh}^2(\operatorname{csh}_1^{-1} x) = \frac{1}{\operatorname{csh}^2(\operatorname{csh}_1^{-1} x)},$$

i.e.:

$$(2) \operatorname{ch}^2(\operatorname{csh}_1^{-1} x) - 1 = \frac{1}{x^2}.$$

From the equality (2), according to Proposition 3.1, point 37) and to the equality (2.2), obtain the equality (5.30), and, according to Proposition 3.1, point 38) and the same equality (2.2), obtain the equality (5.30').

27) According to the first equality from (4.22), for every $x \in \mathbf{R}$,

$$(1) \operatorname{cth}^2(\operatorname{sh}^{-1} x) - 1 = \frac{1}{\operatorname{sh}^2(\operatorname{sh}^{-1} x)} \\ = \frac{1}{x^2},$$

i.e.:

$$(2) \operatorname{cth}^2(\operatorname{sh}^{-1} x) = 1 + \frac{1}{x^2} \\ = \frac{1 + x^2}{x^2}.$$

From the equality (2) and Definition 2.4, obtain that:

$$(3) \operatorname{th}^2(\operatorname{sh}^{-1} x) = \frac{x^2}{1 + x^2};$$

so, the equality (5.31) holds.

28) According to the first equality from (4.21), for every $x \in \mathbf{R}$,

$$(1) 1 - \operatorname{th}^2(\operatorname{ch}^{-1} x) = \frac{1}{\operatorname{ch}^2(\operatorname{ch}^{-1} x)} \\ = \frac{1}{x^2},$$

i.e.:

$$(2) \operatorname{th}^2(\operatorname{ch}^{-1} x) = 1 - \frac{1}{x^2} \\ = \frac{1 - x^2}{x^2}.$$

From the equality (2) and Definition 2.3, obtain the equality (5.32).

29) According to the equalities (2.3) and (2.4), for every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$(1) \operatorname{th}(\operatorname{cth}^{-1} x) = \frac{1}{\operatorname{cth}(\operatorname{cth}^{-1} x)} \\ = \frac{1}{x};$$

therefore, the equality (5.33) holds.

30) According to the second equality to (4.21), for every $x \in (0, 1)$,

$$(1) 1 - \operatorname{th}^2(\operatorname{sch}^{-1} x) = \operatorname{sch}^2(\operatorname{sch}^{-1} x) \\ = x^2,$$

i.e.:

$$(2) \operatorname{th}^2(\operatorname{sch}^{-1} x) = 1 - x^2.$$

From the equality (2) and Proposition 3.1, point 34), obtain the equality (5.34), and from Proposition 3.1, point 35), obtain the equality (5.34').

31) According to the equality (4.22'), for every $x \in (-\infty, 0)$,

$$(1) \operatorname{cth}^2(\operatorname{csh}_1^{-1} x) - \operatorname{csh}^2(\operatorname{csh}_1^{-1} x) = 1,$$

i.e.:

$$(2) \operatorname{cth}^2(\operatorname{csh}_1^{-1} x) = 1 + x^2.$$

From the equality (2) and Definitions 2.3 and 2.4, it follows that:

$$(3) \frac{1}{\text{th}^2(\text{csh}_1^{-1}x)} = 1+x^2,$$

whence, according to Definition 2.3 and to Proposition 3.1, point 37), obtain the equality (5.35). On the other hand, according to the equality (4.22'), for every $x \in (0, +\infty)$,

$$(4) \text{cth}^2(\text{csh}_2^{-1}x) - \text{csh}^2(\text{csh}_1^{-1}x) = 1,$$

i.e.:

$$(5) \text{cth}^2(\text{csh}_2^{-1}x) = 1+x^2.$$

From the equality (2) and Definitions 2.3 and 2.4, it follows that:

$$(6) \frac{1}{\text{th}^2(\text{csh}_2^{-1}x)} = 1+x^2,$$

whence, according to Definition 2.3 and to Proposition 3.1, point 38), obtain the equality (5.35').

32) According to the first equality from (4.22), for every $x \in \mathbf{R}^*$,

$$(1) \text{cth}^2(\text{sh}^{-1}x) - 1 = \frac{1}{\text{sh}^2(\text{sh}^{-1}x)}$$

$$= \frac{1}{x^2},$$

i.e.:

$$(2) \text{cth}^2(\text{sh}^{-1}x) = 1 + \frac{1}{x^2}$$

$$= \frac{1+x^2}{x^2}.$$

From the equality (2) and to Definition 2.4, obtain the equality (5.36).

33) According to the first equality from (4.21), for every $x \in (-\infty, -1) \cup (1, +\infty)$,

$$(1) 1 - \text{th}^2(\text{ch}^{-1}x) - 1 = \frac{1}{\text{ch}^2(\text{ch}^{-1}x)}$$

$$= \frac{1}{x^2},$$

i.e.:

$$(2) \text{th}^2(\text{ch}^{-1}x) = 1 - \frac{1}{x^2}$$

$$= \frac{x^2 - 1}{x^2}.$$

From the equality (2) and Definition 2.4, it follows that:

$$(3) \frac{1}{\text{cth}^2(\text{ch}^{-1}x)} = \frac{x^2 - 1}{x^2},$$

i.e.:

$$(4) \text{cth}^2(\text{ch}^{-1}x) = \frac{x^2}{x^2 - 1},$$

whence obtain the equality (5.37).

34) According to the equalities (2.3) and (2.4), for every $x \in (-1, 1)$,

$$(1) \text{cth}(\text{th}^{-1}x) = \frac{1}{\text{th}(\text{th}^{-1}x)}$$

$$= \frac{1}{x};$$

therefore, the equality (5.38) holds.

35) According to the second equality to (4.21'), for every $x \in (0, 1)$,

$$(1) \text{th}^2(\text{sch}^{-1}x) + \text{sch}^2(\text{sch}^{-1}x) = 1,$$

i.e.:

$$(2) \operatorname{th}^2(\operatorname{sch}^{-1}x) = 1 - x^2.$$

From the equality (2) and Definitions 2.3 and 2.4, it follows that:

$$(3) \frac{1}{\operatorname{cth}^2(\operatorname{sch}^{-1}x)} = 1 - x^2,$$

i.e.:

$$(4) \operatorname{cth}^2(\operatorname{sch}^{-1}x) = \frac{1}{1 - x^2}.$$

From the equality (4) and Proposition 3.1, point 34), obtain the equality (5.39), and from Proposition 3.1, point 35), obtain the equality (5.39').

36) According to the equality (4.22'), for every $x \in (-\infty, 0)$,

$$(1) \operatorname{cth}^2(\operatorname{csh}_1^{-1}x) - \operatorname{csh}^2(\operatorname{csh}_1^{-1}x) = 1,$$

i.e.:

$$(2) \operatorname{cth}^2(\operatorname{csh}_1^{-1}x) = 1 + x^2.$$

From the equality (2) and Definitions 2.3 and 2.4, according to Proposition 3.1, point 37), obtain the equality (5.40). On the other hand, according to the equality (4.22'), for every $x \in (0, +\infty)$,

$$(3) \operatorname{cth}^2(\operatorname{csh}_2^{-1}x) - \operatorname{csh}^2(\operatorname{csh}_1^{-1}x) = 1,$$

i.e.:

$$(4) \operatorname{cth}^2(\operatorname{csh}_2^{-1}x) = 1 + x^2.$$

From the equality (4), Definitions 2.3 and 2.4 and Proposition 3.1, point 38), obtain the equality (5.40').

Bibliography

- Abramowitz, M. & Stegun, I. A. (Eds.) (1973), *Hyperbolic Functions* §4.5 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 83-86.
- Anderson, J. W., (1999), *Trigonometry in the Hyperbolic Plane* §5.7 in Hyperbolic Geometry, New York: Springer-Verlag, p. 146-151.
- Beyer, W. H., (1987), *Hyperbolic Function* CRC Standard Mathematical Tables, 28th ed. Boca Raton, FL: CRC Press, p. 168-186 and 219.
- Harris, J. W. & Stocker, H., (1998), *Hyperbolic Functions*, Handbook of Mathematics and Computational Science, New York, Springer-Verlag, p. 245-262.
- Jeffrey, A., (2000), *Hyperbolic Identities*, §3.5 in Handbook of Mathematical Formulas and Integrals, 2nd ed. Orlando, FL: Academic Press, p. 117-122.
- Yates, R. C., (1953), *Hyperbolic Functions*, A Handbook on Curves and Their Properties. Ann Arbor, MI, J. W. Edwards, p. 113-118.
- Vălcan, D., (2016), *Teaching and Learning Hyperbolic Functions (I); Definitions and Fundamental Properties*, PedActa, Vol. 6, (2016), Nr. 2, p. 1-21.
- Vălcan, D., (2020), *Teaching and Learning Hyperbolic Functions (II); Other Trigonometric Properties and Their Inverses*, Journal of Education and Human Development, Vol. 8, No. 4, 2019, pp. 159-176.
- Zwillinger, D. (Ed.), (1995), *Hyperbolic Functions*, §6.7 in CRC Standard Mathematical Tables and Formulae. Boca Raton, FL, CRC Press, p. 476-481.