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Visualization in Solving Inequality Questions: Case of Pre-Service Mathematics Teachers

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Abstract

The purpose of this case study is to determine whether pre-service teachers use visualisation in solving inequality questions, then, to determine the role of visualisation used as an appropriate visual representation tool in the lessons in resolving the negative aspects that pre-service teachers have. In addition, their views about the use of visual aids in solving problems are investigated. The sample of this study consists of 20 pre-service mathematics teachers who are enrolled in Education Faculty. As data collection tools, a test with open-ended questions was subjected to pre-service teachers. Then, their views were gathered by means of their written responses. At the end, two face to face semi-structured interviews were performed. The data gathered in this study revealed that pre-service teachers generally preferred algebraic solutions, and they have both positive and negative views about using visualisation in solving questions.

Key Words: visualisation, visual representations, inequalities, pre-service mathematics teachers

1. Introduction

Pedagogical Content Knowledge (PCK), which is an important concept related to teacher education, was first introduced by Shulman in 1986. This concept has been revised by different researchers and remains up-to-date. Although the concept of "pedagogical content knowledge" was not mentioned before Shulman's studies, previous studies in the mathematics education area were conducted in line with the components of PCK proposed by (Shulman, 1986).

The components of PCK that teachers require to have were investigated, in general, in five basic sub-headings. These are (a) knowledge of students' understanding, which includes students' difficulty in understanding the concepts taught, (b) knowledge of multiple representations of concepts, (c) knowledge of instructional methods and strategies, (d) knowledge of assessment of students' learning of subject matter, and (e) orientation to teaching subject matter in the curriculum (Akkoç, Özmantar and Bingölbali, 2008). These components may be seen independently from the others. However, they have a close relation with each other and with the Knowledge of the Subject Matter (KSM), which was proposed by Shulman (1986) as another type of knowledge that teachers are required to have.

The concept of visualisation (and/or visual learning) has been discussed for a long time. Especially after the mid-20th century, visualisation became one of the main research areas in mathematics education studies (Presmeg, 2006). Bishop (1989) asserts that visual presentation is very helpful, especially for the indications of the abstract nature of mathematics. While dealing with mathematical concepts, students need to consider things which do not exist physically.

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They are dealing with symbolic representation of questions and trying to give meaning to algebraic (or symbolic) equations. Even in a simple question, they have to base their thoughts on abstract concepts like axioms and theorems (Ferrara, Pratt and Robutti, 2006). In regard to this feature of mathematics, students have a great need to cross from the abstraction level of mathematical formulas or solutions to the concrete level (Goldin, 2002). This can be performed with physical constructions or mental activities. At this point, visualisation or visual representation of the same mathematical notions plays a crucial role.

Visualisation in teaching mathematics has become a research area for many researchers (Arcavi, 2003; Presmeg, 1986; Presmeg, 2006; Rösken and Rolka, 2006). Although visualisation is studied by many researchers, there is no precise definition for this term. For example, Guttierrez (1996) gives this definition from his perception as *the kind of reasoning activity based on the use of visual and spatial elements, either mental or physical, performed to solve problems or prove properties,* while Arcavi (2003) gives a brief meaning for visualisation by means of graphs, diagrams, and modes, which is to *develop and stabilize an interaction between people* (students) *and things.* Zimmermann and Cunningham (1991) defined visualisation as *the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding*, Schnotz, Zink, and Pfeiffer (1995) defined visualisation as *the process of transformation from the visual-spatial model to mental structure* and Zazkis, Dubinsky and Dautermann (1996) defined visualisation as *the act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses.* Conversely, Konyalıoğlu (2003) defined it as *the bridge between the experimental world and, thinking and reasoning.*

As it is stated, visualisation takes different meanings in different contexts. Likewise, this study also has a meaning according to its aim. Visualisation is referred to producing and using geometric representations such as diagrams, graphs, figures… in a mathematical question solving processes in this study.

The inequality subject is one of the important subjects that require pre-knowledge of equation subject, the graphical representation of the equations and parts of functions. Therefore, it requires higher skills to deal with inequality problems in which students can use visualisation as an aid. To better understand the importance of the study, there is a need to have a look at the previous literature related to visualisation and how it is perceived by other researchers.

1.1 Theoretical Background

Now, referring to the Arcavi's definition of visualisation, it can be understood that there is a need for "things" to create an interaction with people for visualisation (Arcavi, 2003). Previous research indicates various ways to visualise learning. Some examples of the ways of visualisation can be 2D or 3D physical manipulatives (Toptaş, 2008), diagram, graphs and pictures (Corter and Zahren, 2007; Harel, Selden and Selden, 2006; Konyalıoğlu, 2003; Tanışlı, 2008) technological tools such as dynamic geometry environments (Cabri, GSP etc…) (Hollebrands, 2003) graphing calculators (Kastberg and Leatham, 2005) and other computer software (Tall, 1986).

Actually, Corter and Zahner (2007) mention two different visual representations. These are internal (e.g. mental imagery) and external (e.g. pictures, graphs, diagrams). All of the visual representation methods mentioned above are external. Such external representations can aid development of students' understandings which, then, leads to the internal representation. Konyalıoğlu (2003) separated visualisation into two forms. These are the non-mathematical representations, which are the samples or the forms of mathematical concepts, which are similar in the physical world and mathematical representations that are in the form of graphical representations rooted from the internal structure of mathematics itself.

In this study, the representations mentioned by Konyalıoğlu (2003) and Corter and Zahner (2007) were combined together. Then, the representations were studied according to representation types as mental- mathematical (mental mathematical graphics, diagrams or shapes), mental-non-mathematical (a mental concrete object), physical-mathematical (mathematical graphics or shapes) and physical-non-mathematical (concrete object). If the types of the visualisation is taken into account, any image such as the shape of graph of any equation that appears in mind can be considered as the mental mathematical, while image of a spherical object such as sun can be considered as mental non-mathematical visualisation. Conversely, the graph of this equation on paper is physical mathematical and the sum itself is a physical non-mathematical type of visualisation.

As we know, mathematics is built onto abstract concepts (Bishop, 1989). Although, mathematics students' need to comprehend the abstract concepts and processes (Teisser, 2005), from students' perspective, the abstract nature of mathematics makes it hard for students to learn and comprehensively understand them (Öçal and Yalçın, 2010). To overcome this problem, many researchers indicate that visualisation plays a crucial role (Arcavi, 2003; Corter and Zahren, 2007; Daugherty, 2007; Gutierrez, 1996; Konyalıoğlu, Aksu and Şenel, 2012; Presmeg, 1986; Presmeg, 2006). To be more specific, for example, Corter and Zahren (2007) stated that, by emphasising the external visual representations such as graphs and diagrams, the visual tools support students' memory, attract their attention, provide models for students, and facilitate the correct solution of the problems.

A picture is worth a thousand words. Actually, this proverb has a broad meaning for us. It may be enough to show a picture to represent a story that requires lots of explanation. Likewise, mathematics learning necessitates generalisation of a theory, comprehending abstract concepts (Zodik and Zaslavsky, 2007). Therefore, students expect to find a method that expresses the complex structure of mathematics in a simple way, which means, a way that is worth thousands of explanations, that is, visualisation. There are numerous advantages of using visualisation in mathematics teaching and learning. For example, as an advantage of using visualisation in classroom applications, it is evident that while students see different representation types at the same time, they can easily make a connection and they can see the effect of any change in one representation type on another one (Jung, 2002). Another advantage mentioned by Hacısalihoğlu (1998) states that the main source of gathering information is the sense of seeing. According to what we see in any object, we can have information about it. Therefore, visualisation may provide students with better understanding of mathematics and mathematical concepts that have a complex and abstract nature. At this point, Yenilmez and Şan (2008) emphasise the importance of pictures, figures and observation of the samples that may provoke mental operations like constructing relations among the abstract concepts. Also, Işık and Konyalıoğlu (2005) stated that visualisation has a positive effect on both cognitive and emotional response in the process of mathematics education.

One of the various advantages of using visualisation is that it prevents students from memory based learning. In fact, memorisation is not considered to be learning. Soylu (2008) advocates the advances of meaningful learning compared to learning with memorisation. For example, most students believe that it is enough to solve questions in the same way as those in previous exams to succeed at the new ones, namely memorisation is in advance. However, without comprehensive understanding of the knowledge necessary for the next exam, students face difficulty in solving new questions that they have never seen before (Abramovitz, Berezine, Berman and Shavartsman, 2009; Ellerton and Clements, 2011). Like Laughbaum (2003), many researchers give evidence that students can comprehend mathematical concepts and subjects by means of visual aids. To provide meaningful learning of the operations based on mathematical concepts and of the relationships among them, teachers should use appropriate methods (Soylu, 2008), which basically include visual reasoning and visual aids. This is because meaningful understanding firstly necessitates the comprehension of such mathematical concepts.

There is evidence that graphical representation of mathematical problems motivates students to deal with it, which is stated by Kreminski (2009). He indicates that students benefit from the visual tools used in learning mathematics concepts by constructing multiple representations for them and linking mental images with such representations. Another positive aspect of visualisation is that students can easily accommodate the procedures needed in successful problem solving processes, and they can transfer the knowledge obtained from previous experience to new situations, and become successful in the new tasks or problems at the end (Hsieh and Lin, 2008). In addition, students can develop their problem solving abilities and their understanding of the mathematical concepts via visualisation (Hitt, 2011). He also asserts that visualising mathematics is a way to transform the abstract nature of mathematics into the concrete, which helps students to comprehend the concepts easily. All in all, using visual aids to teach and learn mathematics subjects is necessary. The methods to use visualisation, however, are gaining importance. If not appropriate, using visualisation may not have a positive effect on students' learning or it may make it worse.

While most studies make assertions about the positive aspects of visual tools and their use in learning and teaching mathematics, some researchers warn people about the possible mis-usage or the wrong use of visual manipulatives. For example, Corter and Zahren (2007) emphasise that there should be a match between the structure of the problem given to students and the visual tool for the usefulness of the visual tool.

If not, there may arise a problem in students' understanding and knowledge construction. Another important point to mention is that students may depend only on the given model or visual material or device (Ponte, Matos, Guimar, Leal and Canavarro, 1994) instead of the knowledge that is aimed to be learnt. It is necessary for students to understand the relationship between the symbolic form of the mathematical notion and its visual representation (Viholainen, 2008). If not, there will be an inverse effect on students' understanding for the introduced knowledge. Zazkis, Dubinsky and Dautermann (1996) briefly explain this situation as *perhaps the most harmful, yet quite common difficulty with visualisation is that students have shown a lack of ability to connect a diagram with its symbolic representation, a process some authors consider to be an essential companion to visualisation.* Eisenberg and Dreyfus (1991) also contribute to this assertion. There is a belief that visual models and representations cannot be a part of proof. In addition, these models and representations are difficult to construct and comprehend from students' perspectives.

Furthermore, Yerushalmy and Chazan (1990) mention possible misleading features of using visualisation. The possible obstacles of using diagrams can be the particularity of the diagrams; such as standard diagrams that result in inflexible thinking that keeps students from understanding and recognition of a concept and that one diagram can be perceived differently. To support the last assertion, Zodik and Zaslavsky (2007) state that students may face difficulty in turning their attention from part of a diagram to the whole one. They may not manage to see the diagram's parts and the relations among them in a whole diagram.

Among mathematicians, there is a prejudgment on visualisation. Since most of the successes in formal mathematics happened via symbolic studies in the 19th and 20th centuries, visual approaches in mathematics are disgraced (Cunningham, 1991). Therefore, students, teachers and other mathematicians think that mathematics requires symbolic and formal illustrations, not visual representations. Therefore, even when encouraged, students and teachers prefer to use formal mathematics when they are asked to solve questions or to show proof (Vinner, 1989).

Looking from the other side, teachers may use visual tools to enhance students' learning. However, students may be engaged more in the diagram or figure used rather than the analogical meaning of what the diagram or figure presents. This results in a situation where students perceive visual tools as the ultimate goal instead of as means to enhance their learning (Zimmermann and Cunningham, 1991). Another aspect is that using visualisation represents only a restricted part of the given condition, and heavy reliance on visualisation may prevent students from mathematical thinking (Brown, 1969; Fennema, 1972; Presmeg, 1986).

Briefly, the literature supports the usage of visualisation, although, at the same time, it warns the students and teachers to be aware of situations that may create misconceptions and misunderstanding of visual aids. Taking those aspects mentioned above into consideration, the importance of this study appears because it gives evidence about the levels of pre-service mathematics teachers' pedagogical content knowledge and that of the subject matter. The visualisation approach can be performed in different ways. In this study, the representation approach was used by means of graphics and shapes. Looking from this perspective, visualisation can be considered to be a tool to help reach the goal of intuiting the easiness of using geometry for solving algebraic questions (Feferman, 2000; Konyalıoğlu, 2003; Konyalıoğlu, 2009).

The foresight of this study is, with the choice of appropriate representation, that visualisation provides students with correctly shaped algebraic processes and the logic behind them (Fischbein, 1999). In addition, visualisation can make students intuitively understand the properties of algebraic processes, which are missed during the learning process and the reasons of the errors that they made (Stavy et all., 2006). In other words, it is aimed to relate the visualisation, which is performed with the mental and physical mathematical representation such as shape, graphics, diagram, with mathematics' fundamental principles and concepts in accordance with the logical consistency between them (Satavy and Tirosh, 2000). Therefore, this helps students to realise some properties missed during learning in algebraic processes.

1.2 Purpose of the Study

The purpose of this study is to determine whether pre-service teachers apply visualisation in solving questions related to inequalities, then, to show the role of visualisation, which was processed with appropriate visual representation in resolving pre-service mathematics teachers' existing negative aspects. In addition, their thoughts about the use of visualisation is attempted to be determined.

These negative aspects are considered to be the instruction based attitudes based on memorisation for students without understanding of algebraic processes, missing some properties about the mathematical subject during the operations and, therefore, making errors in the solution process. The purpose of this study (the distant goal) is to determine how pre-service teachers shape pedagogical content knowledge and knowledge of mathematics, especially about inequalities. Therefore, knowledge of how pre-service teachers shape their own mathematics knowledge would give evidence for how the necessary training should be given in order for effective teaching of mathematics.

The significance of this study is, in accordance with its purpose and limitations, that the confirmation or negation of the expected assumptions of this study will support the negative and positive views on visualisation mentioned in the beginning of the study by means of concrete data.

The reasons why this subject was chosen are students' difficulties in solving problems related to inequalities, the easiness of constructing mathematical visual representations and the variedness of multiple representations. In addition, pre-service teachers were chosen as the sample because they will be the future's teachers. They have to consider both concepts and operations, and the relations between them with a logical frame. In addition, different than students, they have to understand the subject and the concepts in detail. Furthermore, they must have the ability to solve questions with almost no mistakes.

2. Method

2.1 Study Sample

This study was conducted with 20 pre-service mathematics teachers who were in their last grade in the department of Secondary School Mathematics Education in Education Faculty in University. Among them, 11 pre-service teachers were male and 9 were female. Participants of this study were selected among the voluntary students. Before the study, they were informed about the purpose of the study. In addition, their permission was verbally gathered to use the data collected from their responses to interviews and open-ended questions.

Considering the Turkish curriculum for mathematics education, the inequality subject is taught to 9th grade students. Therefore, pre-service teachers' backgrounds are adequate for solving questions related to inequalities, because pre-service teachers have knowledge of the subjects taught in the 9th grade while entering this department. In addition, they have taken Calculus I and Calculus, which necessitates the knowledge of inequality subject.

2.2 Research Design

Case study investigates the factors (environment, individuals, events, processes, etc…) related to a case with who list approach and focuses on how these factors affect the related case and how these factors are affected by it (Yıldırım and Şimşek, 2008). Taking this statement into account, since this study focuses on the effects of using visual representations on resolving pre-assumed pre-service mathematics teachers' negative thoughts, the most appropriate qualitative approach in this study is considered to be the case study. According to Yin (2003), there are three types of case studies, which are exploratory, descriptive and explanatory. The last one deals with "how" and "why" questions related to the case studies. Therefore, the expected goals of this study coincide with this type of study due to its nature.

2.3 Data Collection Tools and Data Analysis

According to Şimşek and Yıldırım (2008), more than one data collection tool should be used for case studies. Therefore, to satisfy the validity and reliability of this study, the triangulation method was used and the data was gathered with more than one data collection tool. With triangulation, researchers can improve the consistence and accuracy of their implications and judgment by means of different kinds of data related to the same phenomenon (Jick, 1979).

In this study, students, firstly, were subjected to a test including open-ended questions, then, their opinions about visualisation application were collected in written form. Then, they were interviewed twice to obtain detailed understanding of their solutions and opinions. The first and second interviews included different contents, although they are directly related to each other in context. The time interval between the two interviews was a week. The interviews were performed with students who agreed to be interviewed.

İnequality test: In the study, it was aimed to determine whether or not pre-service teachers use visualisation in solving questions related to inequalities with the help of a test including open-ended inequality questions. While solving these questions, students were instructed not to solve the questions by using derivatives. The studies of Hızarcı and Elmas (2005) and Yağcı (2009) guided us to develop questions for this test. The questions related to the inequalities subject were about finding interval and finding maximum-minimum values. The questions asked in this test were such that pre-service teachers can easily solve them by using the basic definitions and theorems about inequalities and that they can easily construct graphics for given statements. However, these questions necessitate the knowledge of what the related basic definitions and theorems really mean and to which conditions they can be applied to questions. Otherwise, the attempts to solve questions based on memorisation of such definitions and theorems may result in erroneous results. In line with the pre-service teachers' answers to questions related to inequalities, the solution strategies were categorized. The questions are given below:

1. For $x \epsilon R$ and $-2 < x \leq 3$, find the interval for the expression $\frac{1}{X}$.
2. For $x \epsilon R$ and $-1 < x < 5$, what is the minimum integer value for the expression $x^2 - 2x + 3$?
3. For $x \epsilon R$ and $1 < x < 4$, how many integers are there for the expression $x + \frac{1}{x}$?

Pre-service teachers' opinions in written form: Students were asked questions to gain information about pre-service teachers' general opinions on using visualisation in mathematics education. Their written responses also expect some information about whether they use visual representations such as diagrams, pictures, graphics, concrete objects or mental images, and about whether they construct mental representation in solving questions. They were expected to write their opinions. According to pre-service teachers' responses to these questions, semi-structured interviews were shaped and conducted in order to reach their opinions in detail.

Interview 1: The semi-structured interview including question about pre-service teachers' positive and negative opinions on visualisation and about whether or not they construct internal – external or mathematical – non-mathematical representations was performed with them. First of all, the data was audio-taped and then transcribed into written form. By using the transcribed versions of interviews, the necessary coding process was performed in accordance with content analysis. Each researcher analysed the interviews separately and created the codes. Then, a consensus among the researchers was expected.

Interview 2: With the second interviews with pre-service teachers, some situations were investigated. These situations include whether pre-service teachers who correctly solved questions explain the algorithms logically in the solutions. In addition, the effect of visualisation on explaining the meaning to this algebraic process was also investigated. Furthermore, whether pre-service teachers who could not correctly solve the questions determine the missing points in the solutions was investigated with the help of visualisation. According to these situations, the reason that pre-service teachers operated such algebraic operations in the solution process was investigated in order to indicate their understanding-intuiting situations. In addition, pre-service teachers were expected to discover the missing points and properties in the solutions by using visual representations such as graphics and number lines while solving inequality questions that were parallel to the questions asked in the Inequality Test. Therefore, the effects of visualisation both on logically explaining the mathematical relations and on overcoming the negative aspects such as errors made during the solution process were attempted to be determined with these interviews. The questions in the semi-structured interviews were prepared according to pre-service teachers' solutions to questions in the test. During the interviews performed, necessary notes were taken according to pre-service teachers' answers. With the help of the notes taken, the data was analysed descriptively.

3. Findings

During the data analysis, first of all, among the pre-service teachers who correctly solved questions, the achievement levels of those who used visualisation in any phase of the solution process and who did not use visualisation were presented descriptively. Then, their opinions were gathered in written form in order to gain their opinions about visualisation, whether they use visual representations and whether they construct a mental representation. This guided us to prepare the questions for both interviews.

Therefore, the descriptive data that exists for given answers to the inequality test can be considered in different dimensions. Therefore, this resulted in new interpretations for the data gathered.

3.1 The Findings for the Inequality Test

The table below (Table 1) shows the frequencies of pre-service teachers' correct answers to questions related to inequalities according to the strategies chosen in the solution process.

Table 1: Frequency Table for Solutions Strategies for Solving Inequalities

QUESTIONS	Algebraic(f)	Numeric(f)	Geometric	
			Graphics(f)	Number Line (f)
1T	0	0	0	0
1F	17	0	0	3
2T	3	4	3	0
2F	8	1	1	0
3T	0	1	1	1
3F	15	1	0	1

Note: The T and F next to the question numbers indicate the True and False solutions, respectively.

The majority of the pre-service teachers (17 pre-service teachers) tried to solve the first question algebraically and made a mistake. Some of them gave the answer as $(-\infty, +\infty)$. Conversely, one pre-service teacher indicated the solution as $(-\infty, +\infty)-\{0\}$, although he did not remove the -1/2 point, which makes the expression indefinite. Although three pre-service teachers who used number lines did not miss zero, they could not correctly express the solution set.

Among 20 pre-service teachers, four tried to solve the second question by drawing graphics. Three found the correct answer, although one pre-service teacher could not solve it correctly because he drew the graphics incorrectly. 11 pre-service teachers approached the second question algebraically. Among them, three pre-service teachers found the correct answer by transforming the expression into Absolute Square**.** Four of the five pre-service teachers who approached the second question numerically found the correct answer. These pre-service teachers found the correct answer by trying the integer values in the interval.

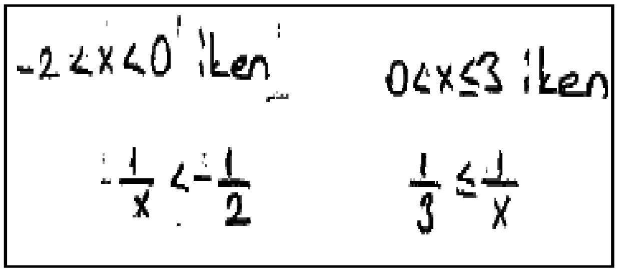
15 pre-service teachers tried to solve the third question algebraically and made a mistake. While a pre-service teacher who approached the third question numerically found the correct answer by using the limit values in the interval, another tried to substitute integer values in the interval to the expression, but could not find the correct answer. Three pre-service teachers constructed a geometric approach to this question. One pre-service teacher who preferred graphics gave the correct answer. Conversely, only one of the two pre-service teachers who used number lines found the correct answer.

The pre-service teachers' reasons for the mistakes made were, basically, that they do not know the types of conditional inequality and equalities for the equations or that of absolute inequalities for numbers or that they confuse them with each other. Therefore, pre-service teachers tried to apply the theories related to absolute inequalities on the conditional inequalities without taking the variable values into account. Therefore, they made mistakes. Especially, the majority of the pre-service teachers who preferred the algebraic and numeric solutions did not consider the inequalities that involve x variable as a function. Conversely, although those who used visual representation intuitively saw the existence of function, they did not directly think of the possible operations for the functions to apply.

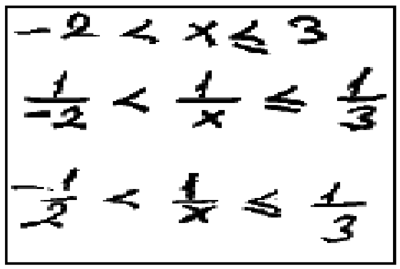
Since pre-service teachers, in general, perceive such inequality expressions as absolute inequality, which is defined by the inequalities for equation expressions or numbers, they did not think that these expressions behave like functions. Therefore, they did not pay attention to the vital property of functions, which is the solution set, and to what conditions are necessary to apply the operations for functions. Although this fosters the usage of multiple representations (NCTM, 2000), it is a reflection on prejudgments, especially for visualisation's misleading effect and to inadequacy of using multiple representations. Some of the Pre-Service Teachers' (PST) answers were as follows:

For question 1:

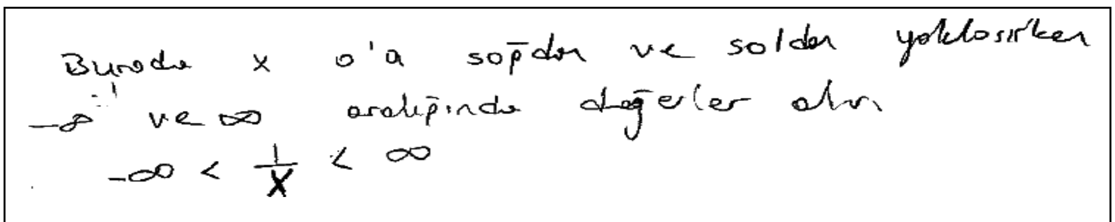
Answer of PST1:



Answer of PST2:

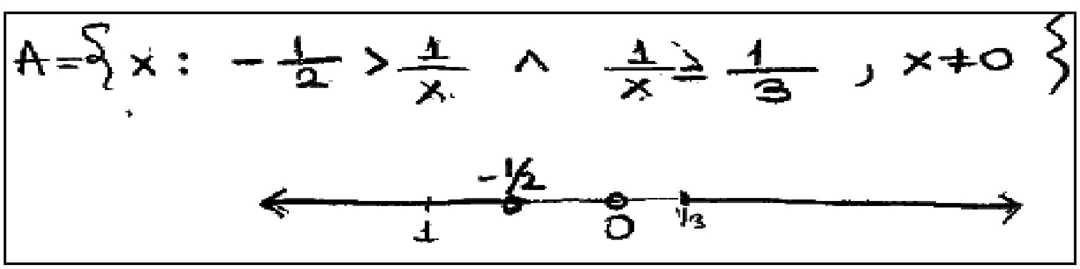


Answer of PST3:



English version: Here, while x approaches to "0" from left and right, it takes values between -∞ and ∞.

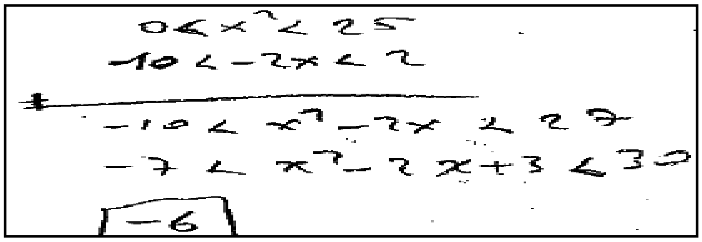
Answer of PST4:



Pre-service teacher PST1 tried to reach the result by breaking the interval into pieces with the thought that the "0" value makes the value of the expression "1/x" indefinite. Conversely, PST2 thought the inequality to be absolute inequality and tried to find the solution by using theorems related to absolute inequalities. Due to a single indefinite point in the expression, this pre-service teacher considered it a limit question. Depending on this thought, although PST3 correctly found the interval from -∞ to ∞, this pre-service teacher did not consider the values in the interval that makes "1/x" indefinite. Lastly, although PST4, who used number line found the correct solution, she expressed the solution set incorrectly.

For question 2:

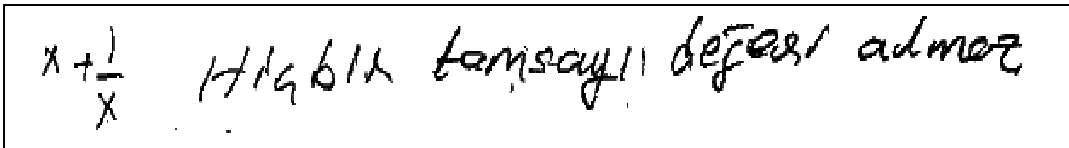
Answer of PST5:



Pre-service teacher PST5 thought the inequality as absolute inequality and tried to solve it by using the absolute inequality theorem. PST5 did not consider the expressions "x^2" and "-2x" as function, therefore, he summed them up without considering the domains.

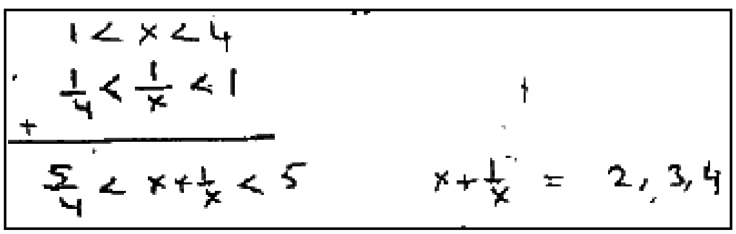
For question 3:

Answer of PST6:



English version: The expression does not take any integer value.

Answer of PST7:



Since PST6 thought that x can take only integer values, she stated that the value of the expression cannot take an integer value. In addition, PST7 thought that the inequality is absolute inequality; he tried to approach the solution by using the absolute inequality theorem. He did not consider the expressions "x" and "1/x" as function; he also added the expressions without considering the domains of such expressions.

3.2 The Findings of Written Opinions

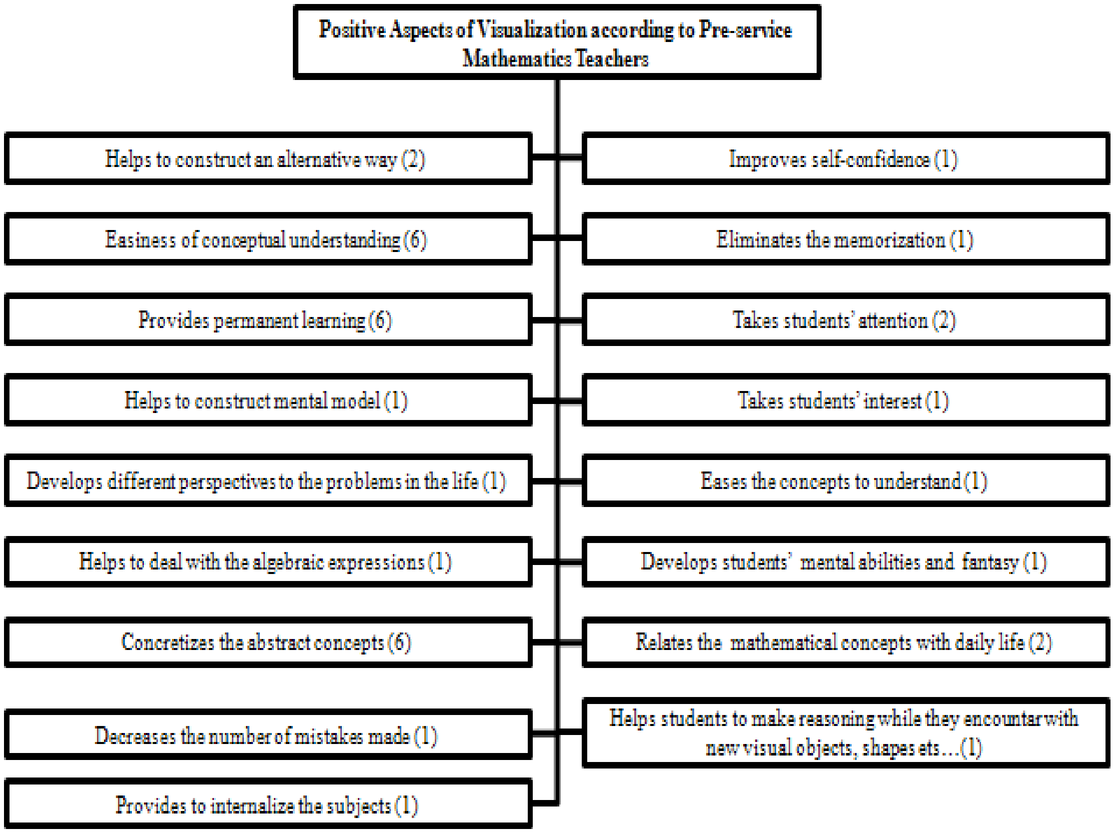
It was found from the data that pre-service teachers have both positive and negative opinions about visualisation. They also stated that they generally do not use visual representations while solving questions. However, the majority of them stated that they construct mental representations related to solutions of the questions. Since the written opinion findings were investigated in detail in Interview 1 and Interview 2, pre-service teachers' opinions were not represented here.

3.2.1 The findings of interview 1

These interviews were performed with 11 pre-service teachers who accepted the voice recordings. The data gathered from Interview 1 was presented in two ways. First of all, their positive and negative opinions about visualisation in mathematics were presented in the diagrams (Figure 1 and 2). Secondly, Table 2 shows the frequencies of whether pre-service teachers construct internal or external representations. However, the majority of students state that according to the subject, usage of visual representation changes.

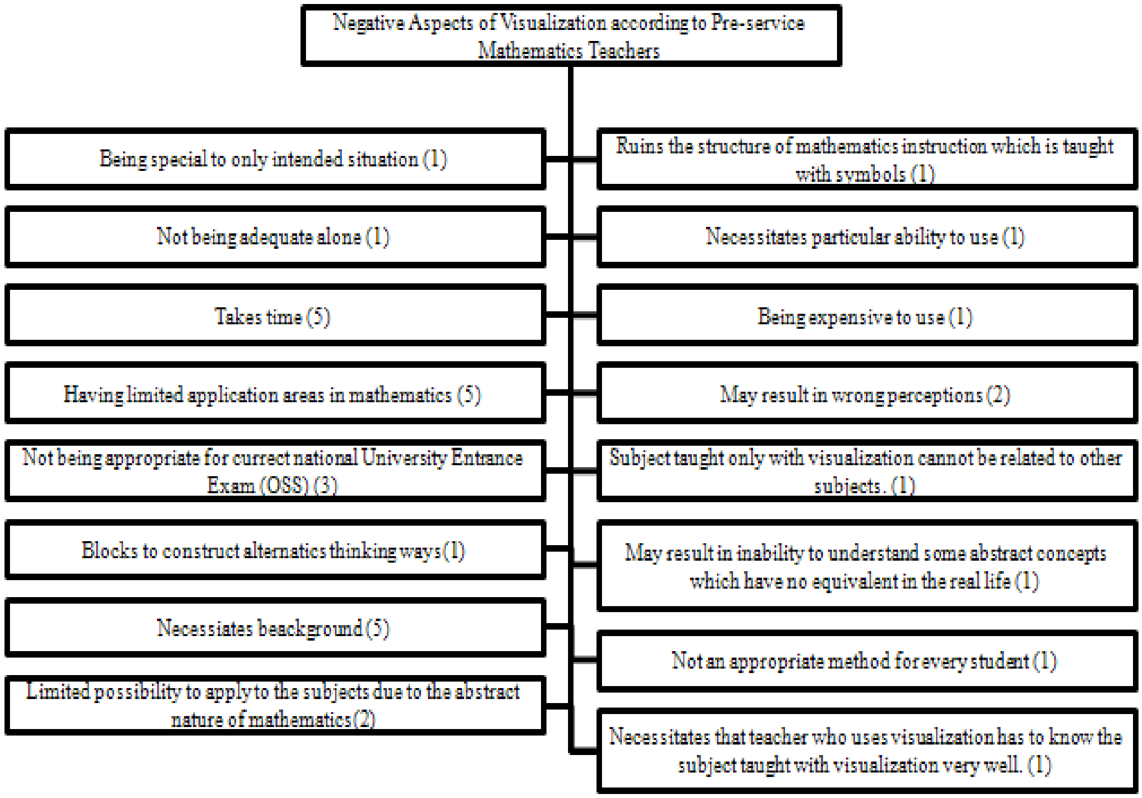
Table 2: Pre-Service Teachers' Usage of Visual Representation

VISUAL REPRESENTATION			
INTERNAL		EXTERNAL	
Mathematical(f)	Non-mathematical(f)	Mathematical(f)	Non-mathematical(f)
5	5	1	-



Note: Since there were only 11 pre-service teachers in the interviews, the percentages were not presented. The numbers inside the parentheses indicated the frequencies.

Figure 1: Positive Aspects of Visualisation according to Pre-service Mathematics Teachers



Note: Since there were only 11 pre-service teachers in the interviews, the percentages were not presented. The numbers inside the parentheses indicated the frequencies

Figure 2: Negative Aspects of Visualisation according to Pre-service Mathematics Teachers

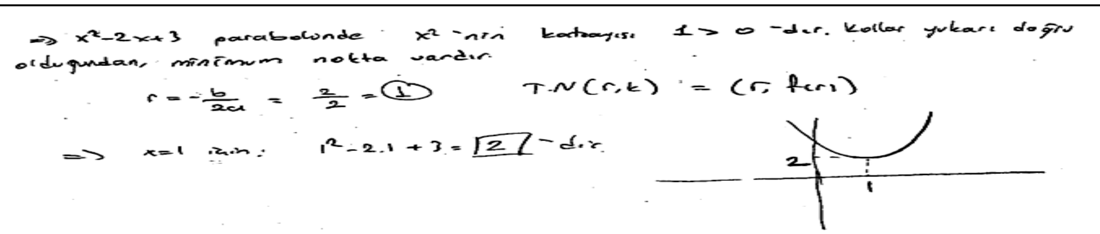
3.2.2 The Findings of interview 2

Interview 2 was conducted with four pre-service teachers who correctly solved the second question and three pre-service teachers who correctly solved the third question. Therefore, seven interviews were performed. Here, Ger, Gir, Ao, No stand for pre-service teachers who used Geometric external representation, Geometric internal representation, Algebraic operation and Numeric operation, respectively. These pre-service teachers' answers to the questions asked were as follows:

A. Intuiting the Reason for the Correct Operation (2nd Question)

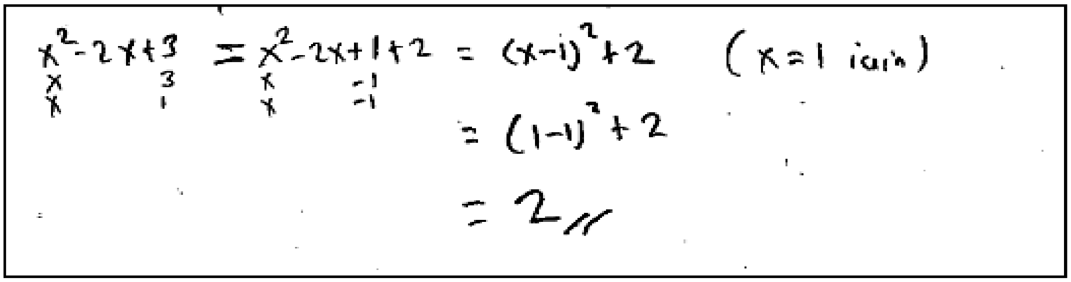
For $x \epsilon R$ and $-1 < x < 5$, what is the minimum integer value for the expression $x^2 - 2x + 3$? Students' answers were indicated below.

Answer of PST8:

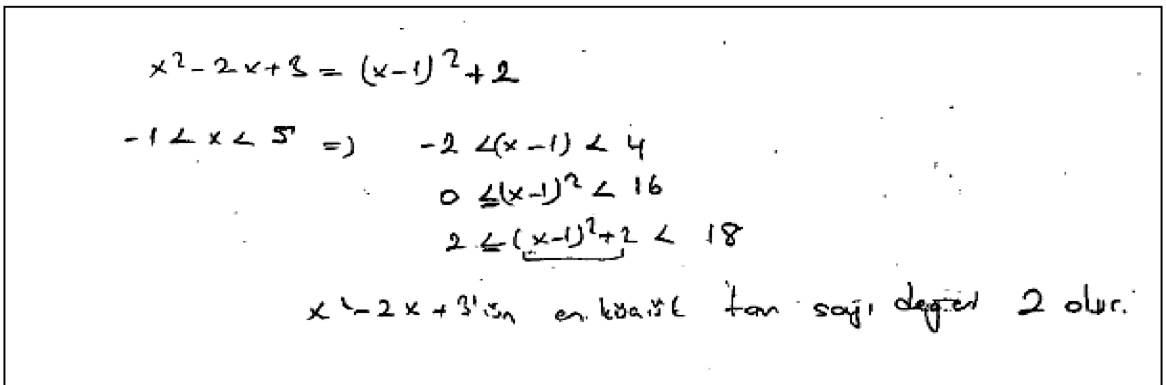


English version: The coefficient of x^2 in the parabola of x^2-2x+3 is 1>0. Since the parabola is concave up, there is a minimum point.

Answer of PST9:

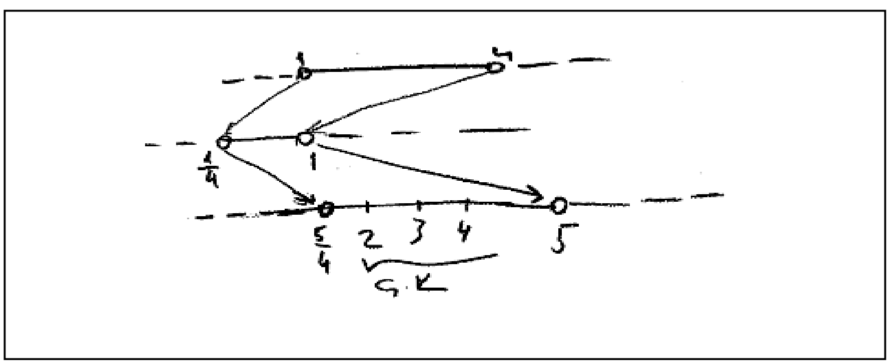


Answer of PST10:



English version: The minimum integer value for x^2-2x+3 is 2.

Answer of PST11:



1. You have the answer sheet here; do you really believe that you solved it correctly?

PST8 (Ger): *Yes, absolutely. There cannot be any other graphics for this.*

PST9 (Gir): *Correct, because I used equalities in the expressions. There is no exception for these expressions. If there is an equation in the expression, it is absolutely correct.*

PST10 (Ao): *Is it possible that it is incorrect? (The researcher stated that she did not imply that). It does not seem incorrect. I guess.*

PST11 (No): *I did it with numbers. The expression is first increasing, then, decreasing. It is absolutely true.*

2. Why did you solve it in this way?

PST8 (Ger): *Since I always think about the equation, inequality and their graphics together, the parabola came to my mind first, while I saw x^2-2x+3. I drew. It was easy. When I see quadratic equations, immediately the parabola appears in my mind.*

PST9 (Gir): *I looked to see whether I can find the root. First of all, I thought about the parabola, but I did not draw it. Then, this method seemed easy to me. I wanted to think about it independent of x. This is because if we transform the expression into a perfect square, it becomes zero. For getting rid of x. You know. We are transforming the peak point to a perfect square and finding the solution.*

PST10 (Ao): *I remembered it. It was said before that we have to solve it in this way. It is easier and more certain. To reach the result easier. Namely, it seemed easier to do it instead of find the intervals for $x^2$and -2x, and adding 3 to them. Due to diminishing the possibility of operational errors. It seemed easier to see it.*

PST11 (No): *Since the integer values are less in this interval, I just tried. Absolutely, the correct answer can be found.*

3. Why did you not find the intervals separately for each term, then sum them up?

PST8 (Ger): *I did not think in this way. Since the parabola came to my mind, I did not think about it separately.*

PST9 (Gir): *It lasts longer. I believed that I can definitely find the result with a perfect square. I did not think it. Maybe it works.*

PST10 (Ao): *As I said, it lasts longer to find the intervals one by one. A perfect square is easier, maybe I make operational mistakes in the other way.*

PST11 (No): *I do not know the reason. This is guaranteed.*

4. Can you do the operation by finding the intervals for each term and summing them up?

Pre-service mathematics teachers did these operations and they found -6 as the result.

5. Now, the result changed. Which one is true for you?

PST8 (Ger): *I do not understand, why did it happen in this way? Is it possible that the inequalities cannot be summed? But, my solution is correct.*

PST9 (Gir): *I missed something, I guess. I do not understand how it happened.*

PST10 (Ao): *I think I made an operational error. Otherwise, did I sum the small equals with small one? I guess it is due to that.*

PST11 (No): *I do not know why. But it is not possible that the expression gets -6 values in this interval. Let's give value to it.*

6. Can you draw the graphics for x^2 and -2x separately? Please mark the limit values.

Pre-service teachers did what was required.

7. Do you know what axes in the coordinate system mean for functions?

The correct answers were gathered.

8. Can you perform addition, again, using the graphics?

PST8 (Ger): *I understood, I added different values of x algebraically. X is 0 in the one of them, and is 5 in another. Of course, all x values should be equal. Similar to the addition operation in the functions.*

PST9 (Gir): *My mistake is adding different values in the function at the same time. It is somehow memorisation. If I drew the graphics and thought accordingly, I could understand clearly.*

PST10 (Ao): *Yes, of course. When I add them separately, I added different values of x. I saw it in the graphics. When I drew them separately, I realised it easily.*

PST11 (No): *I never thought in this way. Ok, but there is no need to do it. Here, it is obvious that there are few numbers of integers. It lasts longer with graphics. But it is good, different. Actually, I know these.*

9. So, why is your solution correct?

PST8 (Ger): *It is guaranteed with parabola, because there is only one expression and graphics. The values of x are the same. They do not change. When drawing the graphics separately, it is obvious.*

PST9 (Gir): *Since my solution is in the common parenthesis directly, mistakes cannot occur, because the values of x are the same. Actually, they cannot be different in this situation.*

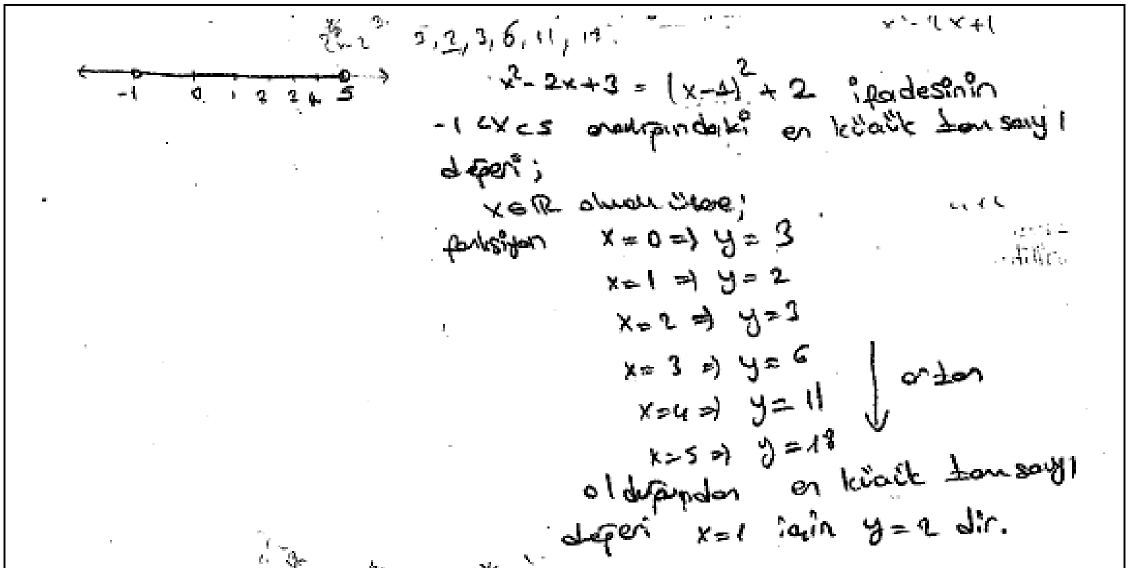
PST10 (Ao): *Actually, my solution is different form of parabola. Actually, I was trying to find the solution with the help of visualisation.*

PST11 (No): *The numeric did not made me make a mistake. Since I gave numeric values, I, actually, did not think separately. But always x values are used. We cannot give 1 and 3 to the term at the same time. Actually, the graphics correct this.*

B. Intuiting the Mistakes They Made (Third Question)

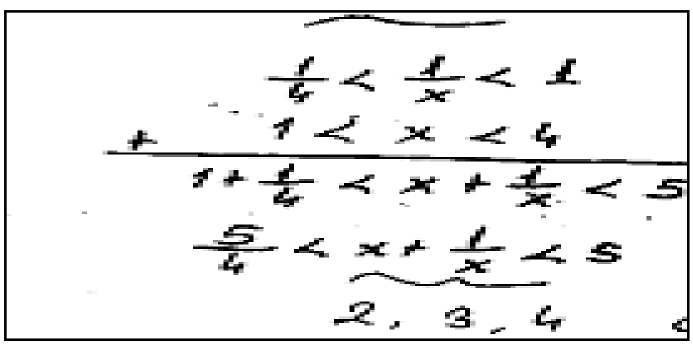
For $x \epsilon R$ and $1 < x < 4$, how many integer values does the expression $x + \frac{1}{x}$ get?

Answer of PST12:

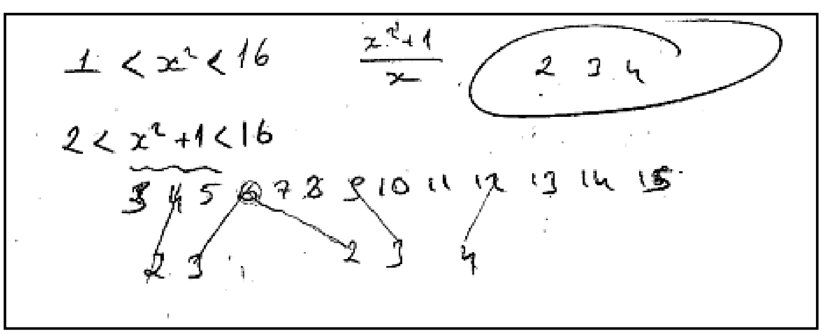


English version: The minimum value of x^2-2x+3=$(x-1)^2$+2 for -1<x<5;
For x∈R; the function ….. Since it is increasing, the minimum integer value for x=1 is y=2.

Answer of PST13:



Answer of PST14:



1. You have the answer sheet here, do you really believe that you solved it correctly?

PST12 (Ger): *Yes. I used number line. I gave values equivalently.*

PST13 (Ao): *Yes. I turned the inequalities to the opposite side and added them . There is such a property, so there should not be a mistake.*

PST14 (No): *Yes.*

2. Why did you solve it in this way?

PST12 (Ger): *I used two number lines. I looked at the equivalent values and added them together.*

PST13 (Ao): *As I said before, I turned the inequalities to the opposite side and added them. There is such a property.*

PST14 (No): *Yes. I tried one by one. I assumed that the expression is increasing and it takes the minimum value when x=1. And it takes the maximum value at 4. For middle values 2, 3 and 4, I took the squares plus 1 and divided it by itself.*

3. Can you draw graphics?

Pre-service teachers experienced difficulty in doing this. Then, the researcher asked the following question:

4. Ok. Can you draw the graphics for the expressions x and 1/x separately?

(Pre-service teachers drew these graphics.)

5. Can you look at the additions again using graphics?

PST12 (Ger): *I took the values outside the interval, too. No… Is it tcorrect? Yes of course…. x is 1 for the first one and 4 for the other. Let me look at it again. Yes, yes. The values of x are different.*

PST13 (Ao): *I did not understand. Can we do it like this? If we do it like this… I understood. But if the variables are different, is my solution ok? Of course, the domains of the functions must be equal for algebraic operations.*

PST14 (No): *I understood. I used trial and error. I may have missed something. This became more scientific…*

6. Can you mark the limit points and the corresponding values in the graphics?

Pre-service teachers did it.

7. Ok. What do you think about your own solution, now?

PST12 (Ger): *But it became in this way in the number line. Yes of course. Mistake… But it is actually the same in the number line, too… I could not understand. I made a mistake. I wish I had tried the graphics.*

PST13 (Ao): *I did not understand why. However, the minimum value of the expression 5/4. That means 2 is in this interval. Are the graphics wrong? But, the graphics never lie. Therefore, this is not like in the equality. I had to add the minimum and maximum. I had to consider the values of x.*

PST14 (No): *The expression can be integer for x's values in the real numbers. Probably, I missed this point. But is what I said correct? I have to think about it.*

4. Conclusion and Discussion

Considering the answers to open-ended questions, the written opinions and interviews in general, it was seen that pre-service teachers who solved the questions correctly could not give logical explanations for the solutions. Similarly, those who made a mistake in solving questions also could not put forward an idea of what their mistakes are and why.

In general, students tried to solve the inequality questions with the algebraic approach and they made a mistake in solving these questions. For example, 17 students tried to solve the first question with the algebraic approach. These findings can be attributed to two explanations. The first is that students generally do not see the visual representation of any solution as formal proof (Alcock and Weber, 2010). Actually, researchers recommend students to base their formal proofs on visual representations such as diagrams and graphics (Raman, 2003) therefore; they can better relate the variables in the questions. Another reason could be that students cannot visualise what is expected in the questions graphically or in other visual means (Weber, 2001).

Compared to algebraic and numeric solutions, the percentage of students' correct answers is higher in the geometric approach, which includes a number line and graphics representations. For students who visualise the solutions, their answers are generally correct. Therefore, visual aids are crucial in solving inequality questions. Students can base their solutions on them. Many researchers also support the use of number lines (Reys, Marilyn, Mary and Nancy, 1998), graphics and diagrams (Arcavi, 2003; Corter and Zahren, 2007; Jung, 2002).

From the written responses of the pre-service mathematics teachers, there appeared both positive and negative attitudes towards visualisation in solving inequalities. Some positive aspects of visualisations with high frequencies were "*it helps to construct alternative methods", "easiness of conceptual understanding", "it concretises the abstract concepts", "it relates mathematical concepts to daily life".* They are all supported by other researchers in the previous literature. For example, Goldin (2002) indicates that visualising the questions or problems helps students to find new solution strategies. In addition, using visual representation of inequality expressions is a way that students can concretise the abstract concepts in mathematical questions (Bishop, 1989; Ferrara, Pratt and Robutti, 2006). Furthermore, researchers mention the direct relations between the conceptual understanding and visual representations of such concepts (Tall, 1991).

In the second part, students also mentioned the negative aspects of the visual materials and their use in solving questions. Actually, some of the drawbacks of using visual representations were stated in the previous literature. For example, Zimmermann and Cunningham (1991) indicated that visualisation should not be students' ultimate goal and they should not try to learn what the function of visual mean is. Instead, they should have the necessary background about using such visual means such as diagrams and graphics. Similarly, pre-service teachers also stated that "*it necessitates background*". In addition, similar to pre-service teachers' responses, applications are inappropriate and they take too much time.

Sometimes, excessive use of visualisation in concept teaching in mathematics blocks alternative thinking (Vinner, 1989), which coincides with the pre-service teachers' negative opinions. They stated that "*visualisation sometimes blocks construction of alternative thinking methods"*.

In the semi-structured interviews, students were intuitively finding their mistakes and the correct solution strategies for the questions asked in the inequality questions test. In general, memorisation plays an important role in solving such questions. Students rely on their memorization and make mistakes. During the interviews, they were provided with some key points to explore at what points they made mistake and how they correct the mistakes. During the application, their intuitive cognition helped them. As is well known, intuition is immediate apprehension about what is studied (Roartry, 1967). Instead of full memorisation during the solution (Jung, 2002) and taking more time to solve it (Herman, 2007), they immediately give meaning to their mistakes and solution strategies (Fischbein, 1987). Therefore, using visual materials helps students intuitively to find the correct answer and they provide students with meaningful understanding.

With the findings of this study, it can be assumed that teachers need to be aware of the positive aspects of using visualisation in solving inequality problems and the possible wrong or misleading usage of visualisation, one of which is considering visual aids as the ultimate goal of learning. In future studies, teachers or pre-service teachers' prejudgments to the use of visualisation in proof processes or in the solutions of the questions can be investigated. Their views and the reasons behind their thoughts can be analysed in detail. Another suggestion for future work is to study the choice of students' use of internal and external visualisations.

This study helped to determine the levels of pre-service teachers' pedagogical and mathematical content knowledge. This can be considered as a diagnosis for the negative aspects of such types of knowledge that pre-service teachers have. Therefore, these negative aspects could be resolved with the help of the findings of this study and corresponding results. More clearly, this study may carry a guideline for the treatment for the insufficiencies and negative aspects of the pedagogical and mathematical concept knowledge that pre-service mathematics teachers have. In the light of the findings of this study, pre-service teachers could not carry their problem solving approaches neither to pedagogical content knowledge nor to mathematical content knowledge.

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